Schottky-like Coverings

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Preface

The objective of this work is to shed some light on $\overline{\mathcal{M}_g}$, the compactification of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g, by considering only the Riemann surfaces with nodes X in $\overline{\mathcal{M}_g}$ which have a given homomorphic image G: Let

$$\mathcal{M}_{g,G} := \left\{X \in \overline{\mathcal{M}_g} : \exists \text{ surjective group hom. } \pi_1(X) \to G\right\}$$

In chapter 1 we will exhibit which groups G need to be considered and prove some combinatorial group theoretic results about them.
To give some understanding of $\mathcal{M}_{g,G}$, we construct an X in the interior of $\mathcal{M}_{g,G}$. The main result of this thesis: If $X \in \mathcal{M}_{g,G}$ and $\varepsilon : \pi_1(X) \twoheadrightarrow G$ is given, the regular covering with $\mathcal{D}eck(Y/X) \underset{\varepsilon}{\xrightarrow{\sim}} G$ is a tree of planar Riemann surfaces. This is proved in chapter 2.
There is some hope that the universal family $\mathcal{X}_{g,G} \xrightarrow{p} \mathcal{M}_{g,G}$ (with $p^{-1}(\{X\}) \cong X$ for every $X \in \mathcal{M}_{g,G}$) is related to the deformation space of G as a Kleinian group. In chapter 3, we show that this deformation space is a finite-dimensional complex analytic manifold and exhibit global holomorphic coordinates for it.

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My thanks also go to the developers of the typesetting packages $\mathcal{AMS}$-TEX and LAMS-TEX, which saved me from the pain of using LATEX.

CONTENTS

Abbreviations

Notations

$\mathbb{N}$	$\{1, 2, 3, \ldots\}$	
$\mathbb{N}_0$	$\{0, 1, 2, 3, \ldots\}$	
$A \hookrightarrow B$	injective mapping	
$A \twoheadrightarrow B$	surjective mapping	
$A \cong B$	A isomorphic to B (w.r.t. some category determined by the context)	
$[a, b]$	commutator $aba^{-1}b^{-1}$ (a, b elements of a group)	
G^{ab}	the abelianization ${}^{G}/_{[G,G]}$ of a group G	
$H \leq G$	H subgroup of a group G	
$N \trianglelefteq G$	N normal subgroup of a group G	
$N_G(U)$	the normalizer of a subgroup $U \leq G$ in a group G	34
$\langle g, h, \ldots \rangle_G$	the normal subgroup of G generated by $g, h, \ldots$	
$\langle g, h, \ldots \rangle_{\trianglelefteq}$	the normal subgroup generated by $g, h, \ldots$	
Δ_g	$\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$	15
$\mathbb{Z}^{*n}$	free group with n generators	15
G^{notor}	${}^{G}/_{\langle \{x \mid \#\langle x \rangle < \infty\} \rangle}$	40
$\mathbb{D}$	the unit disk $\{z \in \mathbb{C} : \lvert z \rvert < 1\}$	25
$\mathbb{H}$	the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$	
$\widehat{\mathbb{C}}$	the Riemann sphere $\mathbb{P}^1(\mathbb{C})$	
$\mathrm{PGL}_2(\mathbb{C})$	the group of Möbius transformations, $= \mathrm{Aut}_{\mathbf{RS}}\left(\widehat{\mathbb{C}}\right)$	
$\mathrm{ir}\,(G)$	inner rank of G	23
$t \preceq t'$	t' finer than t (t, t' partitions)	30
$\gamma_1\gamma_2$	concatenation of γ_1 (first) and γ_2 (second) (γ_1, γ_2 paths)	34
$\pi_1(X)$	fundamental group of X (X path-connected topological space)	15
$\mathrm{Sing}\,(X)$	the set of singularities of X	
$\mathcal{D}eck(Y/X)$	group of covering transformations of the covering $Y \to X$	34
$\mathcal{M}_g$	the moduli space of genus g	41
$\mathcal{M}_{g,G}$	my extended moduli space	42
Mod_g	the modular group of genus g	57
$\mathrm{Diff}(X)$	the group of diffeomorphisms of X	
$f \approx g$	f homotopic to g	56
$v_1 \text{---} v_2$	edge between vertices v_1 and v_2	
$\pi(\mathcal{G})$	fundamental group of a graph of groups	53
$\mathrm{ext}\,(J)$	the exterior of J, i.e. the component of $\widehat{\mathbb{C}} \setminus \overline{J}$ containing ∞	
$\mathrm{int}\,(J)$	the interior of J, i.e. $\widehat{\mathbb{C}} \setminus \overline{\mathrm{ext}\,(J)}$	
$S_f(z)$	the Schwarzian derivative of f at z, see [Nag 88, p. 191-192]	
E_τ	elliptic curve ${}^{\mathbb{C}}/_{\mathbb{Z} \oplus \mathbb{Z}\tau}$ associated to $\tau \in \mathbb{H}$	55
$\mathcal{E}^{-1,1}(X)$	vector space of $(-1,1)$-forms on X, locally $= \varphi(z) \cdot \frac{d\bar{z}}{dz}$	57
$L_\infty^{-1,1}(X)$	$\mathcal{E}^{-1,1}(X)$, equipped with the maximum norm	59

<u>Claim</u>. Underlining of a result means that I haven't found this result in the literature and therefore consider it original work of mine.

0. Introduction and summary of results

0.0. Abstract.

We study a class of coverings of compact Riemann surfaces and stable Riemann surfaces. Specialized to compact Riemann surfaces, it is (mostly) the class of planar coverings, hence a generalization of the Schottky coverings. For stable Riemann surfaces, the covering space is a tree of Riemann surfaces. We also study in detail the deck transformation groups corresponding to these coverings.

0.1. Background.

The first background point is the classification of Riemann surfaces (= connected one-dimensional complex manifolds). The **uniformization theorem** asserts that every Riemann surface X has a universal covering $\tilde{X}$, with $\tilde{X} \cong \mathbb{C}$ or $\tilde{X} \cong \widehat{\mathbb{C}}$ or $\tilde{X} \cong \mathbb{D}$.
It is important to know about the following: If X depends holomorphically on some complex parameters, does the universal covering $p: \tilde{X} \to X$ depend holomorphically on these parameters as well? For example, in the case of compact Riemann surfaces of genus $g = 1$ (elliptic curves), this is known to be true: the elliptic curves are parametrized by $\tau \in \mathbb{H}$, and

$$p: \left\{ \begin{array}{ll} \mathbb{C} \to {}^{\mathbb{C}}/_{(\mathbb{Z}+\mathbb{Z}\tau)} = E_\tau \subset \mathbb{P}^2(\mathbb{C}) \\ z \mapsto \qquad (\wp_\tau(z), \wp'_\tau(z)) \end{array} \right\}$$

is holomorphic in τ.
In order to investigate this more generally, one defines the **moduli space** $\mathcal{M}_g$: the set of compact Riemann surfaces of genus g modulo isomorphy. The above question can then be reformulated as follows: Are there meromorphic functions $p_1(z,\tau), \ldots, p_N(z,\tau)$, defined on $\mathbb{D} \times \mathcal{M}_g$, such that for all $\tau \in \mathcal{M}_g$, the functions $p_1(\cdot,\tau), \ldots, p_N(\cdot,\tau)$ generate the field of meromorphic functions on the Riemann surface X_τ?
An important tool for studying the moduli space are **coverings**, i.e. locally homeomorphic and surjective maps $Y \xrightarrow{p} X$. We will only encounter semi-locally simply connected (hence path-connected) topological spaces. Recall that in this category, to every covering (modulo isomorphism) corresponds a subgroup of $\pi_1(X)$ (modulo conjugacy): $U = p(\pi_1(Y)) \leq \pi_1(X)$, and $U \cong \pi_1(Y)$. We will only need to consider regular coverings, i.e. those with $U \trianglelefteq \pi_1(X)$. In this situation

$$\begin{array}{ccc} \tilde{X} & & 1 \\ \downarrow & & | \\ Y & & \pi_1(Y) \\ \downarrow & & | \\ X & & \pi_1(X) \end{array}$$

p induces an isomorphism $X \cong {}^{Y}/_{\mathcal{D}}$ where $\mathcal{D} = \mathcal{D}eck(Y/X) \cong {}^{\pi_1(X)}/_{U} \cong {}^{\pi_1(X)}/_{\pi_1(Y)}$ is the **deck transformation group**.

Letting X be a Riemann surface of genus g, we have

$$\pi_1(X) \cong \Delta_g := \left\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

The coverings with deck transformation group Δ_g are well-known: They are classified by the Teichmüller space of genus g, defined as

$$T_g = \left\{ \begin{array}{l} (X, \tau) : X = {}^{\tilde{X}}/_{G}, \\ \qquad\qquad \tau :\ \Delta_g \to G \text{ a group isomorphism} \\ \qquad\qquad \text{onto a subgroup } G \leq \mathrm{Aut}_{\mathbf{RS}}\left(\tilde{X}\right) \end{array} \right\} /\cong$$

Clearly

$$\left\{ \begin{array}{rl} T_g & \to \mathcal{M}_g \\ (X, \tau) & \mapsto\ \ X \end{array} \right\}$$

is a covering of $\mathcal{M}_g$. It turns out that T_g is a normal complex space, simply-connected, and $\mathcal{M}_g = {}^{T_g}/_{\mathrm{Mod}_g}$, where $\mathrm{Mod}_g \cong \mathrm{Out}^+(\Delta_g)$ is the so-called **modular group** of genus g.

The coverings with deck transformation group $U = N_{1/2} := \langle a_1, \ldots, a_g \rangle_{\Delta_g} \trianglelefteq \Delta_g$ are well-known as well: they are the **Schottky coverings**. The covering space Y can be embedded into $\widehat{\mathbb{C}}$, $\widehat{\mathbb{C}} \setminus Y$ is a set of measure zero, and every injective holomorphic function $Y \to \widehat{\mathbb{C}}$ is [the restriction of] a Möbius transformation, i.e. an element of $\mathrm{Aut}_{\mathbf{RS}}\left(\widehat{\mathbb{C}}\right) = \mathrm{PGL}_2(\mathbb{C})$. In particular, $\mathcal{D}eck(Y/X) \leq \mathrm{PGL}_2(\mathbb{C})$ is a Kleinian group. The Schottky space, which classifies the Schottky coverings, is

$$S_g = \left\{ \begin{array}{l} (X, \Omega, \varepsilon) : X = {}^{\Omega}/_{G}, \\ \qquad\qquad\quad \Omega \text{ domain in } \widehat{\mathbb{C}}, \\ \qquad\qquad\quad \varepsilon :\ \Delta_g \to G \text{ a surjective group homomorphism}, \\ \qquad\qquad\quad \mathrm{kern}\, \varepsilon = U, \\ \qquad\qquad\quad G \leq \mathrm{Aut}_{\mathbf{RS}}(\Omega), \\ \qquad\qquad\quad \text{hence } \Omega \to X \text{ a Schottky covering} \end{array} \right\} /\cong$$

It is well-known how S_g is related to T_g and $\mathcal{M}_g$:

$$\begin{tikzcd} T_g \arrow[dr, "/\mathcal{H}_g"] \arrow[dd, "/\mathrm{Mod}_g"'] & \\ & S_g \arrow[dl] \\ \mathcal{M}_g & \end{tikzcd}$$

$S_g = {}^{T_g}/_{\mathcal{H}_g}$, where $\mathcal{H}_g$ is the **handlebody group** of genus g.
All of T_g, S_g, $\mathcal{M}_g$ are normal complex spaces of dimension $3g-3$ ($g > 1$).
Their compactifications have already been studied: The elements of $\overline{\mathcal{M}_g}$ are **stable Riemann surfaces** (Riemann surfaces with nodes), the elements of $\overline{S_g}$ are trees of complex projective lines, and T_g can be compactified naively through its embedding into $\mathbb{C}^{3g-3}$. $\overline{\mathcal{M}_g}$ is known to be a projective algebraic variety, $\overline{S_g}$ (see [Gerritzen, Herrlich 88]) is a normal complex space as well. But they correspond only to part of $\overline{T_g}$.

0.2. Objective.

The objective of the present work is to generalize these constructions: Given a group G, we will look at

1. the coverings of X (Riemann surface of genus g) with deck transformation group isomorphic to G,
2. the stable Riemann surfaces X which have a covering with deck transformation group isomorphic to G:

$$\mathcal{M}_g \subset \mathcal{M}_{g,G} := \left\{ \begin{matrix} X : X \text{ stable Riemann surface of genus } g, \\ \text{there is } \varepsilon : \Delta_g \twoheadrightarrow G \end{matrix} \right\} /_{\cong} \subset \overline{\mathcal{M}_g}$$

Of course, we assume here that there are surjective homomorphisms

$$\Delta_g \longrightarrow\!\!\!\!\!\rightarrow G \longrightarrow\!\!\!\!\!\rightarrow {}^{\Delta_g}/_{N_{1/2}} = \mathbb{Z}^{*g}$$

Hence, for a given $X \in \mathcal{M}_{g,G}$ and a surjective group homomorphism $\varepsilon : \Delta_g \twoheadrightarrow G$, we will investigate the covering $Y \longrightarrow X$ which induces $\varepsilon = (\Delta_g = \pi_1(X) \twoheadrightarrow \mathcal{D}eck(Y/X) = G)$.
It is expected that the corresponding space $T_{g,G}$

$$\begin{tikzcd}
T_g \arrow[dddr] & & \\
& T_{g,G} \arrow[dd] & \\
& & S_g \arrow[dl] \\
& \mathcal{M}_g &
\end{tikzcd}$$

is worth investigating, but this topic is not covered by this thesis and is left for future research.

0.3. The groups G.

We will restrict ourselves to those groups G which are the fundamental group of some Riemann surface.

<u>THEOREM 11 (a):</u> The fundamental group $\pi_1(X)$ of a compact Riemann surface with nodes X of genus g is of the form

$$G \cong G_{t,s} := \Delta_{t_1} * \cdots * \Delta_{t_k} * \mathbb{Z}^{*s},$$

with $k \in \mathbb{N}_0$, $t = (t_1, \ldots, t_k) \in {\mathbb{N}_0}^k$, $s \in \mathbb{N}_0$, $r := \sum_{j=1}^{k} t_j$ and $g = r + s$.

We will only consider $G \in \{G_{t,s} : \text{all } t, s\}$ for the following reasons:

- For some part of $\mathcal{M}_{g,G}$, the covering will be exactly the universal covering.
- These are exactly those torsion-free groups for which the covering surface Y of $X \in \mathcal{M}_g$ becomes planar (Theorem 59).

We will prove the following facts about $G = G_{t,s}$.

$$G_{t,s} = \left\langle a_1, b_1, \ldots, a_r, b_r, c_1, \ldots, c_s \mid \prod_{i=1}^{t_1} [a_i, b_i] = 1, \prod_{i=t_1+1}^{t_1+t_2} [a_i, b_i] = 1, \ldots \right\rangle$$

LEMMA 14: There is no surjective group homomorphism $\Delta_g \twoheadrightarrow \mathbb{Z}^{*(g+1)}$.

<u>PROPOSITION 8:</u> The genus g of a CRSN X is

(i) the smallest n such that there is a group homomorphism $\Delta_n \twoheadrightarrow \pi_1(X)$,
(ii) the largest n such that there is a group homomorphism $\pi_1(X) \twoheadrightarrow \mathbb{Z}^{*n}$.

<u>LEMMA 24:</u> $G_{t,s}$ is residually finite.

<u>LEMMA 25:</u> $G_{t,s}$ is hopfian.

<u>NOTE 29:</u> $G_{t,s}$ is torsion-free.

<u>PROPOSITION 28:</u> A stable Riemann surface X with $\pi_1(X) \cong \Delta_g$ cannot have nodes (i.e. is smooth).

PROPOSITION 39 (M. STOLL): $G_{t,s} \cong G_{t',s'}$ if and only if $s = s'$, $k = k'$, and $t = (t_1, \ldots, t_k)$ and $t' = (t'_1, \ldots, t'_{k'})$ are permutations of each other.

PROPOSITION 44 (M. STOLL): Given $G_{t,s}$ and $G_{t',s'}$ with $g = g'$, there is a surjective homomorphism $G_{t,s} \twoheadrightarrow G_{t',s'}$ if and only if $s' \geq s$ and $\big(t'_1, \ldots, t'_{k'}, \underbrace{1, \ldots, 1}_{s'}\big)$ is a finer partition of g than $\big(t_1, \ldots, t_k, \underbrace{1, \ldots, 1}_{s}\big)$.

0.4. The Kleinian groups G.

<u>THEOREM 67':</u> Every $G_{t,s}$ can be realized as covering transformation group of a compact Riemann surface. There is an embedding $G_{t,s} \cong G \leq \mathrm{PGL}_2(\mathbb{C})$ such that:

(i) The **region of discontinuity** Ω of G (which is, by definition, an open subset of $\widehat{\mathbb{C}}$) is non-empty, i.e. G is a **Kleinian group**.
(ii) G is purely loxodromic, if all $t_j > 1$.
(iii) There is a distinguished G-invariant component Ω' of Ω such that a fundamental domain for Ω'/G can be constructed as the intersection of the following:
- a hyperbolic $(4t_1)$-gon,
- $\ldots$,
- a hyperbolic $(4t_k)$-gon,

with $2s$ disks removed.
(iv) G is geometrically finite.
(v) $X = \Omega'/G$ is a compact Riemann surface of genus $g = r + s$.
(vi) $\mathcal{D}eck(\Omega'/X) \cong G \cong G_{t,s}$.

The number of (connected) components of Ω is

$$\left\{\begin{array}{ll} 1 & \text{if all } t_j = 1 \\ 2 & \text{if } k = 1 \text{ and } t_1 > 1 \\ \infty & \text{if } k > 1 \text{ and some } t_j > 1 \end{array}\right\}$$

The number of invariant components of Ω is

$$\left\{\begin{array}{ll} 1 & \text{if } k > 1 \text{ or all } t_j = 1 \\ 2 & \text{if } k = 1 \text{ and } t_1 > 1 \end{array}\right\}$$

<u>THEOREM 116:</u> $T(G,\Omega)$, the deformation space of G, is

$$T(G,\Omega) = T(G,\Omega') \times \prod_{t_j>1} T\big(G_j, \Omega_j''\big)$$

Here all Ω_j'' are simply connected, hence their deformation spaces are simply connected complex manifolds. The deformation space of Ω' is a complex manifold as well. It follows that $T(G,\Omega)$ is a complex manifold of (complex) dimension $3(g-1+r-k)$. $(g \geq 1)$

<u>THEOREM 117:</u> There are $N = 3(g-1+r-k)$ quadruples $(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4})$ of pairwise distinct points $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4} \in \Lambda(G)$ such that

$$\Phi : \left\{\begin{array}{lc} T(G,\Omega) \to & \mathbb{C}^N \\ \quad\mu \quad \mapsto & (\mathrm{DV}\,(w^\mu(x_{i,1}), w^\mu(x_{i,2}), w^\mu(x_{i,3}), w^\mu(x_{i,4})))_{i=1,\ldots,N} \end{array}\right\}$$

is a biholomorphic homeomorphism onto a region $\subset \mathbb{C}^N$. $(g \geq 2)$

0.5. The G-coverings.

<u>THEOREM 75:</u> For every $X \in \mathcal{M}_{g,G}$, there is a regular covering of Riemann surfaces with nodes

$$\begin{tikzcd} Y \arrow[d, "p"'] \\ X \end{tikzcd}$$

with $\mathcal{D}eck(Y/X) \cong G$. The components of Y are planar Riemann surfaces and form a *tree* of Riemann surfaces, i.e. the intersection graph G_Y is a tree, and

$$\pi_1(Y) = \mathop{\ast}_{Y_j \text{ component of } Y} \pi_1(Y_j)$$

<u>THEOREM 84:</u> G_Y is the universal cover of $\mathcal{G}_X$. Here $\mathcal{G}_X$ is the intersection graph of X as a **graph of groups**.

1. The covering groups

1.1. Conventions.

CONVENTION 1:

(a) A **Riemann surface (RS)** is non-empty and connected. A **union of Riemann surfaces (URS)** is a topological space whose connected components are RS.
(b) A morphism of URS is a holomorphic (= analytic) map.

DEFINITION 2:

(a) A **Riemann surface with nodes (RSN)** – a.k.a. stable Riemann surface – is a topological space X, such that every point has a neighbourhood which is either homeomorphic to an open disk $\{z \in \mathbb{C} : |z| < 1\}$ or to the pinched union of two open disks:
$$\{(z,w) \in \mathbb{C}^2 : |z| < 1, |w| < 1, z \cdot w = 0\}$$
– we call these the one or two **neighbourhood disks** of the point –, together with a holomorphic atlas, i.e. a set of homeomorphisms
$$\varphi_U : U \to D_U \subset \mathbb{C}$$
for every neighbourhood disk U, such that the comparison maps $\varphi_{U_2} \circ {\varphi_{U_1}}^{-1}$ (U_1, U_2 neighbourhood disks) are conformal at the non-isolated points of $\varphi_{U_1}(U_1 \cap U_2)$, and such that X is orientable. The points with two neighbourhood disks are called the **nodes** of X. The set of nodes of X is written $N_X := \operatorname{Sing}(X)$. X is called orientable if $X \setminus N_X$ is orientable.
See [Gerritzen, Herrlich 88, § 1.1].
(b) A **morphism of RSN** is a continuous mapping $f : X \to Y$, such that for every point $x \in X$, every neighbourhood disk U of x and every neighbourhood disk U' of $f(x)$ the map $\varphi_{U'} \circ f \circ {\varphi_U}^{-1}$ is holomorphic at the non-isolated points of $\varphi_U\big(U \cap f^{-1}(U')\big)$.

PROPOSITION 3: The RSN form a category; the RS form a full subcategory.

Note that a RSN is in general *not* a manifold.

DEFINITION 4: For a **compact Riemann surface (CRS)** X, the **genus** g is the unique integer ≥ 0 such that its homology group $H_1(X,\mathbb{Z}) = \pi_1(X)^{\mathrm{ab}}$ is isomorphic to $\mathbb{Z}^{2g}$.

DEFINITION 5:
$$\Delta_g := \left\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$
denotes the fundamental group $\pi_1(X)$ of a CRS of genus g.
$$\mathbb{Z}^{*n} := \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ factors}} = \langle c_1, \ldots, c_n \rangle$$
denotes the free group with n generators.

1.2. Decomposing a RSN.

The following decomposition technique for a **compact Riemann surface with nodes (CRSN)** X will often be used:
Let N_X be the (finite!) set of nodes of X. The closure (in X) of each component of $X \setminus N_X$ is called a **component of** X. Let $X_1, \dots, X_k$ be the components of X.

(I) Let X_j' be X_j with blown-up singularities (replace each node by two points, one for each neighbourhood disk). X_j' is a CRS. Note that every holomorphic function on X restricts to a holomorphic function on X_j and therefore gives rise to a holomorphic function on X_j'.

(II) The **intersection graph** G_X of X is a finite undirected graph, having the vertex set $\{X_1, \dots, X_k\}$ and an edge between X_i and X_j $(i \neq j)$ for every node belonging to both X_i and X_j. (Nodes connecting an X_j to itself are discarded here; they have already been handled in (I).)

Note that this definition differs from the one given in [Gerritzen, Herrlich 88, § 1.1].

Definition 6: For a CRSN X, the **genus** $g(X)$ is defined by decomposition of X:

(I) $g(X_j) := g\big(X_j'\big) + \#\{\text{nodes of } X_j\}$

(II) Define the **chromatic number** of a nonempty connected graph G to be

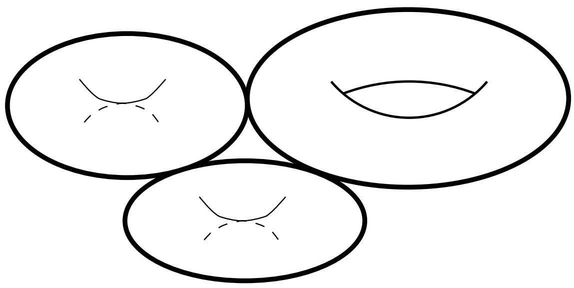
$$\chi(G) := \#\{\text{edges of } G\} - \#\{\text{vertices of } G\} + 1 \in \mathbb{N}_0$$

and put

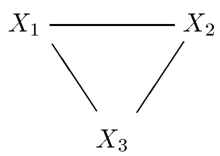
$$g(X) := \sum_{j=1}^{k} g(X_j) + \chi(G_X)$$

Example 7:

$X =$



$G_X =$



Here we have $g(X_1) = 0 + 1$, $g(X_2) = 1 + 0$, $g(X_3) = 0 + 1$ and therefore $g(X) = g(X_1) + g(X_2) + g(X_3) + 1 = 4$.

The following will be proved in Lemma 17, p. 23, and Lemma 18, p. 23:

<u>PROPOSITION 8:</u> The genus g of a CRSN X is

(i) the smallest n such that there is a group homomorphism $\Delta_n \twoheadrightarrow \pi_1(X)$,
(ii) the largest n such that there is a group homomorphism $\pi_1(X) \twoheadrightarrow \mathbb{Z}^{*n}$.

DEFINITION 9: A **stable CRSN (SCRSN)** is a CRSN X, such that for every component X_j of X:

$$2 \cdot g\big(X_j'\big) + \#\{\text{nodes of } X_j\} + \#(N_X \cap X_j) > 2$$

Note that in this term, a node connecting X_j to X_i $(i \neq j)$ is counted once, but a node connecting X_j to itself is counted twice.

In other words, a CRSN X is stable, unless it has some component X_j with $g(X_j) = 0$ and 0 or 1 or 2 nodes, or it consists of a single component which is of genus 1.

1.3. The list of groups.
We now determine presentations for the groups $G = \pi_1(X)$, where X is a CRSN.

<u>DEFINITION 10:</u> For $k \in \mathbb{N}_0$, $t = (t_1, \ldots, t_k) \in {\mathbb{N}_0}^k$, $s \in \mathbb{N}_0$, put $r := \sum_{j=1}^{k} t_j$ and

$$G_{t,s} := \Delta_{t_1} * \cdots * \Delta_{t_k} * \mathbb{Z}^{*s}.$$

<u>THEOREM 11:</u>

(a) The fundamental group $\pi_1(X)$ of a CRSN X of genus g is of the form $G \cong G_{t,s}$, with $g = r + s$. More precisely, k is the number of components X_j of X, $t_j = g\big(X_j'\big)$, and $s = \#N_X - k + 1$.
(b) Every $G_{t,s}$ is the fundamental group of some CRSN. Every $G_{t,s}$ except $G_{0,0}$, $G_{0,1}$, $G_{0,2}$, $G_{1,0}$ is the fundamental group of some SCRSN.

Proof:

(a) By decomposition of X in components X_j. Let $t_j := g\big(X_j'\big) \in \mathbb{N}_0$. Then $\pi_1\big(X_j'\big) \cong \Delta_{t_j}$.
   (I) Let $n_j := \#\{\text{nodes of } X_j\}$.
   CLAIM 1: $\pi_1(X_j) \cong \pi_1\big(X_j'\big) * \mathbb{Z}^{*n_j}$.
   Proof: According to [Schubert 71, III.5.7. Satz 2, p. 206], for every RSN X, $\pi_1(X)$ is isomorphic to the **edge path group** (german: Kantenwegegruppe) of X. Choose a 1-dimensional **skeleton** (german: Gerüst) of X_j (as a **cell complex**), e.g. a **simplicial decomposition** (german: simpliziale Zerlegung). (It is understood that the nodes of X_j are vertices of the skeleton.) Then fix a vertex x_0 of the skeleton and consider all combinatorial paths on the skeleton starting at x_0 and ending at x_0 – **combinatorial paths** are words over a set of symbols, one symbol per path element, – modulo the border paths of the 2-dimensional surface pieces as relations. Now, take a skeleton for X_j' and, for every

pair $\{x_1, x_2\}$ of points which correspond to the same point of X_j, insert an edge between x_1 and x_2. The resulting skeleton has the same edge path group as X_j, as can be seen by collapsing the edge between x_1 and x_2 to a point.

Each inserted edge contributes a free generator and no relations. Therefore $\pi_1(X_j) \cong \pi_1(X'_j) * \mathbb{Z}^{*n_j}$.

(II) Claim 2: $\pi_1(X) \cong \pi_1(X_1) * \cdots * \pi_1(X_k) * \mathbb{Z}^{*\chi(G_X)}$.

Proof: Use the edge path group again. Among the edges E of the intersection graph G_X – recall that each edge corresponds to a node between two different components of X – choose a minimal spanning set, i.e. a set E' of $k-1$ edges that keep the graph connected. By blowing up the singularities $\in E \setminus E'$ (replacing each by two distinct points) one gets a CRSN X' with the same components $X_1, \ldots, X_k$ as X and whose intersection graph $G_{X'}$ is a tree. By the Seifert – van Kampen theorem ([Stillwell 80, Chapter 3.4], [Stöcker, Zieschang 94, Satz 5.3.11.], [Cohen 89, Corollary 2, p. 86]) $\pi_1(X') \cong \pi_1(X_1) * \cdots * \pi_1(X_k)$. Taking a skeleton for X' and adding an edge between x_1 and x_2 for every pair $\{x_1, x_2\}$ corresponding to a node $\in E \setminus E'$, one gets a skeleton which has the same edge path group as X, as can be seen by collapsing the edge between x_1 and x_2 to a point. Each inserted edge contributes a free generator and no relation. Therefore $\pi_1(X) \cong \pi_1(X') * \mathbb{Z}^{*\left(\#\left(E \setminus E'\right)\right)}$.

Summarizing, we have

$$\begin{aligned}
\pi_1(X) &\cong \pi_1(X_1) * \cdots * \pi_1(X_k) * \mathbb{Z}^{*(\#E-k+1)} \\
&\cong \Delta_{t_1} * \cdots * \Delta_{t_k} * \mathbb{Z}^{*\left(\sum_j n_j + \#E - k + 1\right)} \\
&= \Delta_{t_1} * \cdots * \Delta_{t_k} * \mathbb{Z}^{*(\#N_X - k + 1)}
\end{aligned}$$

(b) Obvious.

q.e.d.

Note: Any $t_j = 0$ can be discarded, and the (t_j) can be permuted, without changing $G_{t,s}$. This enables us to henceforth assume that all $t_j \geq 1$ when investigating a particular $G_{t,s}$.

<u>Example 12:</u> For small g, we have the following possible $G_{t,s}$, and at least the following surjective homomorphisms between them. The homomorphisms shown here are built up from $\Delta_{i+j} \twoheadrightarrow \Delta_i * \Delta_j$ and $\Delta_{i+1} \twoheadrightarrow \Delta_i * \mathbb{Z}$. Note that these homomorphisms are by no means unique. It will be shown in Theorem 44, p. 30, that no surjective homomorphisms exist between two different $G_{t,s}$ unless they can be joined by arrows in the diagrams below.

$g = 1$:

$$\begin{tikzcd}
\Delta_1 \arrow[dr, two heads] & \\
& \mathbb{Z}
\end{tikzcd}$$

$g = 2$:

$$
\begin{tikzcd}
\Delta_2 \arrow[d, two heads] & & \\
\Delta_1 * \Delta_1 \arrow[dr, two heads] & & \\
& \Delta_1 * \mathbb{Z} \arrow[dr, two heads] & \\
& & \mathbb{Z}^{*2}
\end{tikzcd}
$$

$g = 3$:

$$
\begin{tikzcd}
\Delta_3 \arrow[d, two heads] & & & \\
\Delta_2 * \Delta_1 \arrow[d, two heads] \arrow[dr, two heads] & & & \\
\Delta_1 * \Delta_1 * \Delta_1 \arrow[dr, two heads] & \Delta_2 * \mathbb{Z} \arrow[d, two heads] & & \\
& \Delta_1 * \Delta_1 * \mathbb{Z} \arrow[dr, two heads] & & \\
& & \Delta_1 * \mathbb{Z}^{*2} \arrow[dr, two heads] & \\
& & & \mathbb{Z}^{*3}
\end{tikzcd}
$$

$g = 4$:

$$\begin{tikzcd}
 & \Delta_4 \arrow[dl, two heads] \arrow[dr, two heads] & & & & & \\
\Delta_2 * \Delta_2 \arrow[dr, two heads] & & \Delta_3 * \Delta_1 \arrow[dl, two heads] \arrow[dr, two heads] & & & & \\
 & \Delta_2 * \Delta_1 * \Delta_1 \arrow[d, two heads] \arrow[drr, two heads] & & \Delta_3 * \mathbb{Z} \arrow[d, two heads] & & & \\
 & \Delta_1 * \Delta_1 * \Delta_1 * \Delta_1 \arrow[drr, two heads] & & \Delta_2 * \Delta_1 * \mathbb{Z} \arrow[d, two heads] \arrow[dr, two heads] & & & \\
 & & & \Delta_1 * \Delta_1 * \Delta_1 * \mathbb{Z} \arrow[dr, two heads] & \Delta_2 * \mathbb{Z}^{*2} \arrow[d, two heads] & & \\
 & & & & \Delta_1 * \Delta_1 * \mathbb{Z}^{*2} \arrow[dr, two heads] & & \\
 & & & & & \Delta_1 * \mathbb{Z}^{*3} \arrow[dr, two heads] & \\
 & & & & & & \mathbb{Z}^{*4}
\end{tikzcd}$$

1.4. A group-theoretical lemma.

LEMMA 13: If $g < h$, there is no surjective group homomorphism $\varepsilon : \mathbb{Z}^{*g} \to \mathbb{Z}^{*h}$.

Proof: ε induces a surjective homomorphism $\bar{\varepsilon}$ between the abelianized groups $(\mathbb{Z}^{*g})^{\mathrm{ab}} = \mathbb{Z}^g$ and $\left(\mathbb{Z}^{*h}\right)^{\mathrm{ab}} = \mathbb{Z}^h$: $\bar{\varepsilon} : \mathbb{Z}^g \to \mathbb{Z}^h$ surjective. Tensoring with $\mathbb{Q}$, one gets a surjective vector space homomorphism $\mathbb{Q}^g \to \mathbb{Q}^h$. Now look at the vector space dimensions. **q.e.d.**

LEMMA 14: There is no surjective group homomorphism $\eta : \Delta_g \to \mathbb{Z}^{*(g+1)}$.

Three proofs are known for this.
Proof 1 ([Haible, 20 September 1994]) is based on the following

LEMMA 15 OF JACO: If $\eta_1, \eta_2 : \Delta_g \to \mathbb{Z}^{*g}$ are surjective group homomorphisms, there is a 3-dimensional closed oriented manifold M (**closed** means compact and without border), such that $M = M_1 \cup M_2$ with M_1, M_2 **handlebodies** (german: Henkelkörper), $X = M_1 \cap M_2$ a CRS of genus g, $\eta_1 = \pi_1(X \hookrightarrow M_1)$, $\eta_2 = \pi_1(X \hookrightarrow M_2)$.

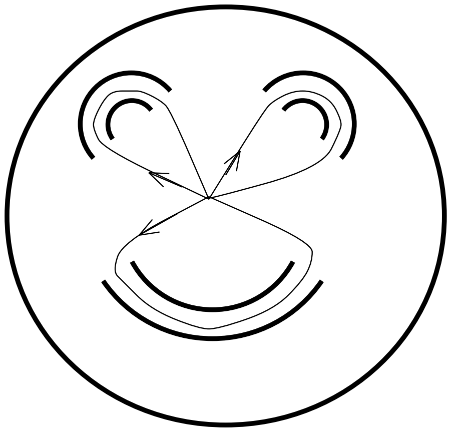
See [Jaco 69, Theorem 5.2, p. 375]. (Attention: M is not embeddable into $\mathbb{R}^3$ in most cases!)
It is clear by the theorem of Seifert and van Kampen ([Stillwell 80, Chapter 3.4], [Stöcker, Zieschang 94, Satz 5.3.11.], [Cohen 89, Corollary 2, p. 86]) that

$$\pi_1(M) = \pi_1(M_1) \underset{\pi_1(X)}{*} \pi_1(M_2)$$

Proof 1 of Lemma 14: We may assume $g \geq 1$. Take a RS X of genus g, i.e. $\Delta_g = \pi_1(X)$. Write $\mathbb{Z}^{*(g+1)} = \mathbb{Z}^{*g} * \langle c \rangle$. This gives a surjective group homomorphism

$$\eta_1 : \Delta_g \xrightarrow{\eta} \mathbb{Z}^{*(g+1)} \xrightarrow{c \mapsto 1} \mathbb{Z}^{*g}.$$

By Jaco's Lemma 15, p. 20, there is a handlebody M_0 with $\partial M_0 = X$ and $\eta_1 = \pi_1(X \hookrightarrow M_0)$. On X, choose a system of curves $\alpha, \beta_1, \dots, \beta_{g-1}$, pairwise disjoint with the exception of the common basepoint, such that $\eta_1([\alpha])$, $\eta_1([\beta_1])$, ... , $\eta_1([\beta_{g-1}])$ are a basis of $\pi_1(M_0)$.



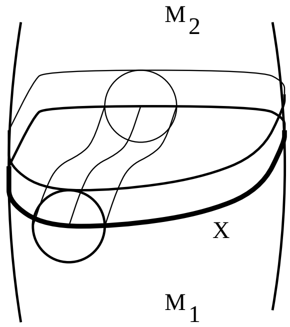
Put $a := \eta([\alpha])$, $b_1 := \eta([\beta_1])$, ... , $b_{g-1} := \eta([\beta_{g-1}])$. We then have $\mathbb{Z}^{*(g+1)} = \langle a, b_1, \dots, b_{g-1}, c \rangle$ (no relators). Together with

$$\eta_1 : \Delta_g \xrightarrow{\eta} \mathbb{Z}^{*(g+1)} \xrightarrow{c \mapsto 1} \mathbb{Z}^{*g}$$

we consider

$$\eta_2 : \Delta_g \xrightarrow{\eta} \mathbb{Z}^{*(g+1)} \xrightarrow{a \mapsto 1} \mathbb{Z}^{*g}.$$

Invoking Jaco's Lemma 15, p. 20, again, there is a 3-dimensional manifold $M = M_1 \cup M_2$, $X = M_1 \cap M_2$, with M_1, M_2 handlebodies and $\eta_1 = \pi_1(X \hookrightarrow M_1)$, $\eta_2 = \pi_1(X \hookrightarrow M_2)$.
We can assume that M is a metric space ([Bröcker, Jänich 73, Bemerkung (7.12), p. 74]). For $\varepsilon > 0$, let $B_\varepsilon(x)$ be the open ball with radius ε around x in M, and $B_i := \overline{\bigcup_{x \in \beta_i} B_\varepsilon(x)}$ the thickened path β_i, $B := B_1 \cup \dots \cup B_{g-1}$ the union of these, and $M_2' := \{x \in M_2 : B_\varepsilon(x) \subset M_2\}$ a somewhat cauterized M_2.



When $\varepsilon > 0$ is small enough, M_2' will be a retract of M_2, i.e. $\pi_1(M_2' \hookrightarrow M_2)$ bijective, and $\pi_1(B) = \pi_1(\beta_1 \cup \cdots \cup \beta_{g-1}) \cong \mathbb{Z}^{*(g-1)}$. By the theorem of Seifert and van Kampen ([Stillwell 80, Chapter 3.4], [Stöcker, Zieschang 94, Satz 5.3.11.], [Cohen 89, Corollary 2, p. 86])

$$\begin{aligned}
\pi_1(M_1 \cup B \cup M_2') &= \pi_1(M_1 \cup B) \underset{\pi_1(B)}{*} \pi_1(B \cup M_2') \\
&= \pi_1(M_1) \underset{\pi_1(B)}{*} \pi_1(M_2') \\
&= \pi_1(M_1) \underset{\pi_1(\beta_1 \cup \cdots \cup \beta_{g-1})}{*} \pi_1(M_2) \\
&= \langle a, b_1, \ldots, b_{g-1} \rangle \underset{\langle b_1, \ldots, b_{g-1} \rangle}{*} \langle b_1, \ldots, b_{g-1}, c \rangle \\
&= \langle a, b_1, \ldots, b_{g-1}, c \rangle
\end{aligned}$$

and $\eta = \pi_1(X \hookrightarrow M_1 \cup B \cup M_2')$. The inclusion chain $X \subset M_1 \subset M_1 \cup B \cup M_2'$ induces a factorization

$$\eta = \underbrace{\pi_1(M_1 \hookrightarrow M_1 \cup B \cup M_2')}_{=:\varepsilon} \circ \underbrace{\pi_1(X \hookrightarrow M_1)}_{=:\delta}.$$

Since η is surjective, $\varepsilon : \mathbb{Z}^{*g} \to \mathbb{Z}^{*(g+1)}$ is surjective as well. But this contradicts Lemma 13, p. 20. **q.e.d.**

Proof 2 of Lemma 14: See [Steinberg 64, Corollary 2, p. 187]. This proof works in the setting of 3-step nilpotent groups, by adjunction of n-th roots of group elements (the analogue to $\mathbb{Q} \underset{\mathbb{Z}}{\otimes}$ in Lemma 13, p. 20). **q.e.d.**

Proof 3 of Lemma 14: The proof in [Shapiro, Sonn 74] uses cohomology of groups (see [Brown 82], [Hilton, Stammbach 71], [Serre 68]). In their setting, $G = \Delta_g$, $n = 2g$, $q = 0$. Proposition 1 [ibidem] shows that the cup product $\cup : H^1(G, \mathbb{Z}) \times H^1(G, \mathbb{Z}) \to H^2(G, \mathbb{Z})$ is a bilinear map corresponding to the matrix

$$\begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix},$$

therefore its rank is $m = 2g > 0$. Theorem 1 [ibidem] proves that any free group being the homomorphic image of G has rank $\leq n - \frac{m}{2} = g$. **q.e.d.**

1.5. Two alternative definitions of the genus.
Given $G_{t,s}$ and $g := r + s$, we clearly have two surjective group homomorphisms

$$\tau : \Delta_g \to G_{t,s}$$

$$\beta : G_{t,s} \to \mathbb{Z}^{*g}$$

With the notation

<u>FORMULA 16:</u>

$$G_{t,s} = \mathop{\ast}_{j=1}^{k} \left\langle a_{j,1}, b_{j,1}, \ldots, a_{j,t_j}, b_{j,t_j} \mid \prod_{i=1}^{t_j} [a_{j,i}, b_{j,i}] = 1 \right\rangle * \langle c_1, \ldots, c_s \rangle$$

τ and β can be described as follows:

$$\tau : \left\{ \begin{array}{ll} \left\{ \begin{array}{l} a_{t_1+\cdots+t_{j-1}+i} \mapsto a_{j,i} \\ b_{t_1+\cdots+t_{j-1}+i} \mapsto b_{j,i} \end{array} \right\} & (1 \leq j \leq k,\, 1 \leq i \leq t_j) \\ \left\{ \begin{array}{l} a_{r+i} \mapsto c_i \\ b_{r+i} \mapsto 1 \end{array} \right\} & (1 \leq i \leq s) \end{array} \right\},$$

$$\beta : b_{j,i} \mapsto 1 \quad (1 \leq j \leq k,\, 1 \leq i \leq t_j).$$

<u>LEMMA 17:</u> g is the smallest n such that there is a surjective group homomorphism $\Delta_n \to G_{t,s}$.

Proof: If $n < g$, we have

$$\Delta_n \twoheadrightarrow G_{t,s} \xrightarrow{\beta} \mathbb{Z}^{*g} \twoheadrightarrow \mathbb{Z}^{*(n+1)}$$

surjective, which contradicts Lemma 14, p. 20. **q.e.d.**

<u>LEMMA 18:</u> g is the largest n such that there is a surjective group homomorphism $G_{t,s} \to \mathbb{Z}^{*n}$.

Proof: If $n > g$, we have

$$\Delta_g \xrightarrow{\tau} G_{t,s} \twoheadrightarrow \mathbb{Z}^{*n} \twoheadrightarrow \mathbb{Z}^{*(g+1)}$$

surjective, which contradicts Lemma 14, p. 20. **q.e.d.**

1.6. The inner rank.
Following [Lyndon, Schupp 77, I.6, p.52], we define:

DEFINITION 19: The **inner rank** of a group G is

$$\mathrm{ir}\,(G) := \sup\,\{n \in \mathbb{N}_0 : \text{there is a surjective group homomorphism } G \to \mathbb{Z}^{*n}\}.$$

In this setting,

LEMMA 13$'$: $\mathrm{ir}\,(\mathbb{Z}^{*g}) = g$.

LEMMA 14$'$: $\mathrm{ir}\,(\Delta_g) = g$.

LEMMA 18$'$: $\mathrm{ir}\,(G_{t,s}) = g$.
The following propositions hold:

LEMMA 20: $\mathrm{ir}\,(G_1 * G_2) = \mathrm{ir}\,(G_1) + \mathrm{ir}\,(G_2)$.

Proof: (See [Lyndon, Schupp 77, I.6, Prop.6.4., p.52].)
"$\geq$" Surjective homomorphisms $\varepsilon_1 : G_1 \to \mathbb{Z}^{*n_1} \hookrightarrow \mathbb{Z}^{*(n_1+n_2)}$ and $\varepsilon_2 : G_2 \to \mathbb{Z}^{*n_2} \hookrightarrow \mathbb{Z}^{*(n_1+n_2)}$ induce a surjective group homomorphism $\varepsilon : G_1 * G_2 \to \mathbb{Z}^{*(n_1+n_2)}$.
"$\leq$" Let $\varepsilon : G_1 * G_2 \to \mathbb{Z}^{*n}$ surjective. Let $F_i := \varepsilon(G_i) \leq \mathbb{Z}^{*n}$. By the Nielsen–Schreier subgroup theorem ("every subgroup of a free group is free", see [Lyndon, Schupp 77, Prop. I.2.11.], [Lyndon, Schupp 77, Prop. I.3.8.], [Lyndon, Schupp 77, Prop. III.3.3.]) F_i is a free group. Since $n_i := \mathrm{ir}\,(G_i) < \infty$, F_i is finitely generated and of rank $\leq n_i$. The existence of a surjective group homomorphism $F_1 * F_2 \to \langle F_1, F_2\rangle = \mathbb{Z}^{*n}$ implies, by Lemma 13, p. 20, that $n_1 + n_2 \geq \mathrm{rank}\,(F_1) + \mathrm{rank}\,(F_2) \geq n$. **q.e.d.**

<u>LEMMA 21:</u> If $N \trianglelefteq \mathbb{Z}^{*n}$ and $N \neq \{1\}$, then $\mathrm{ir}\left(\mathbb{Z}^{*n}/N\right) \leq n - 1$.

Proof: Choose a $\gamma \in N \setminus \{1\}$. If there was a surjective group homomorphism $\varepsilon : \mathbb{Z}^{*n}/N \to \mathbb{Z}^{*n}$, denote by α_i the i-th generator of $\mathbb{Z}^{*n}$ and $\beta_i := \varepsilon(\alpha_i N)$. If some $\beta_i = \beta_j$ $(i \neq j)$, we would have a surjective group homomorphism

$$\mathbb{Z}^{*(n-1)} \hookrightarrow \mathbb{Z}^{*n} \twoheadrightarrow \mathbb{Z}^{*n}/N \xrightarrow{\varepsilon} \mathbb{Z}^{*n},$$

which contradicts Lemma 13, p. 20. Therefore $\#\{\beta_1, \ldots, \beta_n\} = n$, and the fact that a set of n elements of $\mathbb{Z}^{*n}$ which generates $\mathbb{Z}^{*n}$ is already a basis of $\mathbb{Z}^{*n}$ ([Lyndon, Schupp 77, Prop. I.2.7., p. 8], see also [Lyndon, Schupp 77, Prop. I.3.5., p. 14] or Lemma 25, p. 25) implies that $\{\beta_1, \ldots, \beta_n\}$ is a basis for $\mathbb{Z}^{*n}$. This is impossible since $\varepsilon(\gamma) = 1$ is a nontrivial relation in $\{\beta_1, \ldots, \beta_n\}$. **q.e.d.**

<u>LEMMA 22:</u> If $\gamma \in \mathbb{Z}^{*n} \setminus \{1\}$ and $\gamma \in N \trianglelefteq \mathbb{Z}^{*n}$, then $\mathrm{ir}\left(G_{t,s} * \mathbb{Z}^{*n}/N\right) \leq g + n - 1$.

Proof: This follows from the previous Lemmas 20, 18$'$ and 21:

$$\mathrm{ir}\left(G_{t,s} * \mathbb{Z}^{*n}/N\right) = \mathrm{ir}\,(G_{t,s}) + \mathrm{ir}\left(\mathbb{Z}^{*n}/N\right) \leq g + (n-1).$$

q.e.d.

1.7. More about Δ_g.
Recall that a group G is called **residually finite** if

$$\forall h \in G \setminus \{1\} : \quad \exists N \trianglelefteq G : \quad (G : N) < \infty, h \notin N.$$

(See [Lyndon, Schupp 77, Chap. IV.4., p. 195] and [Birman 77, Chap. 6.6, p. 196].)

Lemma 23: Δ_g is residually finite.

Proof: We may assume $g > 0$. Choose a CRS X of genus g. Its universal covering $\mathbb{X}$ is either the unit disk $\mathbb{D}$ or (if $g = 1$) $\mathbb{C}$:

$$\begin{array}{c} \mathbb{X} \\ \Big\downarrow /\Delta_g \\ X \end{array}$$

with $\mathcal{D}eck(\mathbb{X}/X) = \pi_1(X) = \Delta_g$. Choose a fundamental polygon P for the action of Δ_g in $\mathbb{X}$, such as the Dirichlet region [Maskit 88, IV.G.2] or, if $g > 1$, the Ford region [Maskit 88, II.H.3]. $h \in \Delta_g \setminus \{1\}$ corresponds to a path in $\mathbb{X}$, beginning at some $z_0 \in \mathbb{X}$. Cover this path (or, more precisely, some neighbourhood of it) by a simply connected finite set of Δ_g-translates of P. Let Q be the resulting polygon and r its boundary, considered as a word in the generators ${a_1}^{\pm 1}, {b_1}^{\pm 1}, \ldots, {a_g}^{\pm 1}, {b_g}^{\pm 1}$ and adjusted to begin at z_0. Then, by the Poincaré polygon theorem [Maskit 88, IV.H.11], Q is the fundamental polygon of some group $G \leq \Delta_g$ acting discontinuously on $\mathbb{X}$. Let $Y := {}^{\mathbb{X}}\!/_G$ denote the resulting RS. It is obvious that:

(i) $\pi_1(Y) = \langle a_1, b_1, \ldots, a_g, b_g \mid r \rangle$.

(ii) The canonical map $Y = {}^{\mathbb{X}}\!/_G \to {}^{\mathbb{X}}\!/_{\Delta_g} = X$ is an n-sheeted unlimited unbranched covering (see section 2.1), where n is the number of translates of P that constitute Q.

(iii) $(\Delta_g : G) = n < \infty$ (follows from (ii)).

(iv) $h \in \Delta_g$, $h \notin G$.

Finally use a standard trick to descend from G to a normal subgroup $N \trianglelefteq \Delta_g$, $N \leq G \leq \Delta_g$, $(\Delta_g : N) \leq n!$: just define N as the kernel of the action of Δ_g on the cosets ${}^{\Delta_g}\!/_G$:

$$N := \operatorname{kern} \left\{ \begin{array}{rc} \Delta_g \to & S_{{}^{\Delta_g}/_G} \\ \alpha \mapsto & (\beta G \mapsto \alpha\beta G) \end{array} \right\}$$

q.e.d.

Lemma 24: $G_{t,s}$ is residually finite.

Proof: Use Lemma 23, p. 25 and the fact that the free product of finitely many residually finite groups is residually finite, see [Cohen 89, Corollary, p. 27]. **q.e.d.**

Recall that a group G is called **hopfian** if every surjective group homomorphism $G \to G$ is an isomorphism. ([Lyndon, Schupp 77, p. 14])

Lemma 25: $G_{t,s}$ is hopfian.

Proof: This follows from Lemma 24, p. 25 and the fact that finitely generated residually finite groups are hopfian ([Lyndon, Schupp 77, Theorem IV.4.10., p. 197]). **q.e.d.**

Lemma 26: If $k > 1$ and $t_1 > 0, \ldots, t_k > 0$, there is no surjective homomorphism $\Delta_{t_1} * \cdots * \Delta_{t_k} \to \Delta_{t_1 + \cdots + t_k}$.

Proof: Assume we have a surjective homomorphism $\varepsilon : G_{t,0} \to \Delta_r$. Together with $\tau : \Delta_r \to G_{t,0}$ (defined in section 1.5) we have $\varepsilon \circ \tau : \Delta_r \to \Delta_r$ surjective. By Lemma 25, p. 25, $\varepsilon \circ \tau$ is an isomorphism. This is impossible because $\prod_{i=1}^{t_1} [a_i, b_i] \in \mathrm{kern}\,(\tau) \setminus \{1\}$.

q.e.d.

<u>LEMMA 27:</u> A surjective homomorphism $G_{t,s} \to \Delta_g$ ($g = r + s$, all $t_j > 0$) only exists in the trivial case $k \leq 1$, $s = 0$.

Proof: Let $\varepsilon : G_{t,s} \to \Delta_g$ be a surjective homomorphism. ε induces a surjective homomorphism $\bar{\varepsilon}$ between the abelianized groups $(G_{t,s})^{\mathrm{ab}} = \mathbb{Z}^{2t_1+\cdots+2t_k+s} = \mathbb{Z}^{2r+s}$ and $(\Delta_g)^{\mathrm{ab}} = \mathbb{Z}^{2g} = \mathbb{Z}^{2r+2s}$. Tensoring with $\mathbb{Q}$ and looking at the vector space dimensions, we get $2r + s \geq 2r + 2s$, so $s = 0$. $k \leq 1$ now follows from lemma 26, p. 25. **q.e.d.**
We can now prove a partial converse to section 1.3:

<u>PROPOSITION 28:</u> If X is a SCRSN with $\pi_1(X) \cong \Delta_g$, then X is a CRS without nodes.

Proof: By section 1.3, $\pi_1(X) \cong G_{t,s}$. By Lemma 27, p. 26, $s = 0$, and there is at most one component X_j of X with genus $g(X_j) > 0$. $s = 0$ implies:

(i) The components X_j of X are CRS without nodes.
(ii) The intersection graph is a tree.

If there is more than one component, the intersection graph has at least two leaves (vertices belonging to exactly one edge). One of these must be of genus 0, with only one node $\in N_X$ on it; this contradicts the definition of SCRSN. **q.e.d.**

<u>NOTE 29:</u> $G_{t,s}$ is torsion-free.

Proof: This follows from the fact that torsion in free products comes from the free factors ([Lyndon, Schupp 77, Theorem IV.1.6., p. 177]) and the fact that one-relator groups such as Δ_{t_j}, where the relator is not a proper power, are torsion-free ([Lyndon, Schupp 77, Prop. II.5.18., p. 108]). **q.e.d.**

<u>NOTE 30:</u> $G_{t,s}$ is a free group only if $r = 0$. In particular, Δ_g is not a free group for $g > 0$.

Proof: If $\mathbb{Z}^{*n} \cong G_{t,s}$, then the inner rank is $n = g = r + s < \infty$, whereas the rank of the abelianization is $n = 2r + s$. **q.e.d.**

<u>PROPOSITION 31:</u> The group homology of $G_{t,s}$ is

$$H_n(G_{t,s}, \mathbb{Z}) = \left\{ \begin{array}{ll} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}^{2r+s} & \text{for } n = 1 \\ \mathbb{Z}^k & \text{for } n = 2 \\ 0 & \text{for } n > 2 \end{array} \right\}$$

Proof: $n = 0$: For any G-module M with trivial operation, we have $H_0(G, M) = M_G = M$ (see [Brown 82, III.1., (1.5), p. 57]). $n > 0$: For any finite collection (G_ν) of groups, one has the Mayer-Vietoris result $H_n\left(\mathop{\ast}\limits_{\nu} G_\nu, M\right) = \bigoplus\limits_{\nu} H_n(G_\nu, M)$ (see [Brown 82, VII.9., (9.2), p. 179]). Now, the homology of the building blocks of $G_{t,s}$

$$H_n(\mathbb{Z}, M) = \left\{ \begin{array}{ll} M_G = M & \text{for } n = 0 \\ M^G = M & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{array} \right\}$$

(see [Brown 82, III.1., Example 1, p. 58]) and

$$H_n\left(\Delta_{t_j}, \mathbb{Z}\right) = \left\{ \begin{array}{ll} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}^{2t_j} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 2 \\ 0 & \text{for } n > 2 \end{array} \right\}$$

(see [Brown 82, II.4., Example 2, p. 37]) needs only be put together. **q.e.d.**

The following was first proved by Michael Stoll (13 June 1995), using the Euler characteristic of $G_{t,s}$ ([Brown 82, IX.6-7.]).

COROLLARY 32: If $G_{t,s} \cong G_{t',s'}$, then $g = g'$, $r = r'$, $s = s'$, $k = k'$.

Proof: By Lemma 18′, p. 24, we have $g = g'$. Looking at the abelianized group $H_1(G_{t,s}, \mathbb{Z})$, we get $g + r = 2r + s = 2r' + s' = g' + r'$. Looking at $H_2(G_{t,s}, \mathbb{Z})$, we get $k = k'$. **q.e.d.**

1.8. Distinguishing different G.

The results in this section (except the last one) were all proved by Michael Stoll (15 June 1995).

Given $G_{t,s}$ (with $g := r + s$) and $G_{t',s'}$ (with $g' := r' + s'$), we will investigate when $G_{t,s} \cong G_{t',s'}$ and when there is a surjective homomorphism $G_{t,s} \twoheadrightarrow G_{t',s'}$.

The following proposition about the (lower) central series of a group is well-known:

PROPOSITION 33: Let G be a group. Define G_n ($n \in \mathbb{N}$) through $G_1 := G$ and $G_{n+1} := [G, G_n] = \langle [g, h] : g \in G, h \in G_n \rangle$ for $n \in \mathbb{N}$. Then:

(i) $G_n \trianglelefteq G \quad \forall n \in \mathbb{N}$.

(ii) $G_{n+1} \trianglelefteq G_n \quad \forall n \in \mathbb{N}$.

Define groups $F_n := {}^{G_n}\!/_{G_{n+1}} \quad \forall n \in \mathbb{N}$.

(iii) $F_1 = G^{\mathrm{ab}}$.

(iv) F_n is an abelian group $\forall n \in \mathbb{N}$.

(v) There is a surjective group homomorphism

$$\left\{ \begin{array}{ccc} F_1 \otimes F_n & \to & F_{n+1} \\ (g \cdot G_2) \otimes (h \cdot G_{n+1}) & \mapsto & [g, h] \cdot G_{n+2} \end{array} \right\} \quad \forall n \in \mathbb{N}$$

(vi) There is a surjective multilinear form

$$\left\{ \begin{array}{ccc} \bigotimes\limits_{i=1}^{n} F_1 & \to & F_n \\ (g_1 \cdot G_2) \otimes \cdots \otimes (g_n \cdot G_2) & \mapsto & [g_1, [g_2, \ldots [g_{n-1}, g_n] \ldots]] \cdot G_{n+1} \end{array} \right\} \quad \forall n \in \mathbb{N}$$

(vii) There is a surjective alternating multilinear form

$$\left\{ \begin{array}{rcl} \bigwedge^2 F_1 & \to & F_2 \\ (g \cdot G_2) \wedge (h \cdot G_2) & \mapsto & [g,h] \cdot G_3 \end{array} \right\}$$

(viii) If G is finitely generated by N generators, then F_n is finitely generated by at most N^n generators (at most $\binom{N}{2}$ generators for $n = 2$).

(ix) Every group homomorphism between two groups $G' \to G''$ induces homomorphisms $G'_n \to G''_n$ and $F'_n \to F''_n$. Every surjective group homomorphism between two groups $G' \twoheadrightarrow G''$ induces surjective homomorphisms $G'_n \twoheadrightarrow G''_n$ and $F'_n \twoheadrightarrow F''_n$.

(x) Every group isomorphism $G' \cong G''$ induces isomorphisms $G'_n \cong G''_n$ and $F'_n \cong F''_n$.

Proof: Well-known and easy. See [Robinson 82, p. 119-122]. **q.e.d.**

LEMMA 34: For $G = G_{t,s}$ we have $F_1 \cong \mathbb{Z}^{2r+s}$ and $F_2 \cong \mathbb{Z}^{\binom{2r+s}{2}-k}$.

Proof: F_1 is a free abelian group generated by $a_{1,1}, b_{1,1}, \ldots, a_{k,t_k}, b_{k,t_k}, c_1, \ldots, c_s$. The kernel of the map $\bigwedge^2 F_1 \to F_2$ is generated by the k elements $a_{j,1} \wedge b_{j,1} + \cdots + a_{j,t_j} \wedge b_{j,t_j}$. Therefore one gets a $\mathbb{Z}$-basis of F_2 by removing $\left\{a_{j,t_j} \wedge b_{j,t_j} : j \in \{1, \ldots, k\}\right\}$ from the standard $\mathbb{Z}$-basis of $\bigwedge^2 F_1$. **q.e.d.**

COROLLARY 35: If there is a surjective homomorphism $G_{t,s} \twoheadrightarrow G_{t',s'}$, then

(a) $g \geq g'$,

(b) if $g = g'$, then $r \geq r'$, $s \leq s'$,

(c) if $g = g'$ and $s = s'$, then $r = r'$, $k \leq k'$.

Proof:

(a) $g = \mathrm{ir}\,(G_{t,s}) \geq \mathrm{ir}\,(G_{t',s'}) = g'$.

(b) We get a surjective homomorphism ${G_{t,s}}^{\mathrm{ab}} \twoheadrightarrow {G_{t',s'}}^{\mathrm{ab}}$. Consider the ranks of these free $\mathbb{Z}$-modules: $2g - s = 2r + s \geq 2r' + s' = 2g' - s'$.

(c) By Proposition 33(xi), p. 27, and Lemma 34, p. 28, we get a surjective homomorphism $\mathbb{Z}^{\binom{2r+s}{2}-k} \to \mathbb{Z}^{\binom{2r'+s'}{2}-k'}$. Tensoring with $\mathbb{Q}$, we see $\binom{2r+s}{2} - k \geq \binom{2r'+s'}{2} - k'$. By assumption, $r + s = r' + s'$ and $s = s'$, so $r = r'$ and $k \leq k'$.

q.e.d.

LEMMA 36: Every subgroup $U \leq \Delta_g$ is either free (of finite or countable rank), or of finite index in Δ_g, and $U \cong \Delta_h$ with $h = (\Delta_g : U) \cdot (g-1) + 1 \geq g$.

Proof: Let X be a RS of genus g. $U \leq \Delta_g$ corresponds to a covering $p : Y \to X$ of X, by Proposition 48, p. 34. By Proposition 52, p. 35, $\pi_1(Y) \cong U$. If Y is not compact, $\pi_1(Y)$ is free ([Stillwell 80, 4.2.2., p. 143]). If Y is compact, it is a RS of some genus h, and p is an n-sheeted covering. Clearly $(\Delta_g : U) = n$. By Riemann-Hurwitz ([Farkas, Kra 80, I.2.7., p. 21]), we have $h - 1 = n \cdot (g-1)$. **q.e.d.**

PROPOSITION 37: Δ_g is not a proper free product.

Proof: (There is a proof in [Lyndon, Schupp 77, II.5.14., p. 107], but I did not understand it.) Write $\Delta_g = A * B$ and assume $A, B \neq \{1\}$. Of course $g > 0$. A and B are infinite (since Δ_g is torsion-free [Lyndon, Schupp 77, Prop. II.5.18., p. 108]). Therefore A, B are of infinite index in Δ_g, hence free by lemma 36, p. 28, therefore Δ_g must be free, a contradiction to Note 30, p. 26. **q.e.d.**

Let us recall the **Kurosh subgroup theorem** about subgroups of free products:

THEOREM 38 (KUROSH): Let $H \leq \mathop{\ast}\limits_{i\in I} G_i$. Then $H = F * \mathop{\ast}\limits_{j\in J} \left(u_j \cdot G_{i(j)} \cdot {u_j}^{-1} \cap H\right)$ where F is free, J is some set and $u_j \cdot G_{i(j)} \cdot {u_j}^{-1} \cap H$ is a subgroup of a conjugate of $G_{i(j)}$ (depending on j).

(See [Cohen 89, chapter 7, Theorem 8, p. 175], [Lyndon, Schupp 77, Theorem IV.1.10., p. 178] and also [Lyndon, Schupp 77, Prop. III.3.6., p. 120], [Stillwell 80, 4.3.9., p. 166], [Cohen 89, chapter 8.5., p. 212, p. 293].)

PROPOSITION 39: If $G_{t,s} \cong G_{t',s'}$, then $s = s'$, $k = k'$, and $t = (t_1, \ldots, t_k)$ and $t' = (t'_1, \ldots, t'_{k'})$ are permutations of each other.

Proof: Let $\varphi : G_{t,s} \to G_{t',s'}$ be a group isomorphism. For every $i \in \{1, \ldots, k\}$, $\varphi(\Delta_{t_i}) \leq G_{t',s'}$ is a free product by Kurosh's Theorem 38, p. 29. By Proposition 37, p. 28, this product has only one factor, and by Note 30, p. 26, the free factor is trivial. Therefore $\varphi(\Delta_{t_i})$ is contained in a conjugate of one of the factors of $G_{t',s'}$. $\varphi(\Delta_{t_i})$ is not a subgroup of $\mathbb{Z}$ (by Note 30, p. 26, again), so

$$\varphi(\Delta_{t_i}) \leq g'_i \cdot \Delta_{t'_{\sigma(i)}} \cdot {g'_i}^{-1} \tag{1}$$

with some $\sigma(i) \in \{1, \ldots, k'\}$ and $g'_i \in G_{t',s'}$. Symmetrically, for every $j \in \{1, \ldots, k'\}$, we have

$$\varphi^{-1}\left(\Delta_{t'_j}\right) \leq g_j \cdot \Delta_{t_{\tau(j)}} \cdot {g_j}^{-1} \tag{2}$$

with some $\tau(j) \in \{1, \ldots, k\}$ and $g_j \in G_{t,s}$.

Combining both, we have

$$\Delta_{t_i} = \varphi^{-1}(\varphi(\Delta_{t_i})) \leq \varphi^{-1}(g'_i) \cdot g_{\sigma(i)} \cdot \Delta_{t_{\tau(\sigma(i))}} \cdot {g_{\sigma(i)}}^{-1} \cdot \varphi^{-1}(g'_i)^{-1}$$

We must have $\tau(\sigma(i)) = i$ (otherwise consider the homomorphism $G_{t,s} \to \Delta_{t_i}$ which maps all other factors of $G_{t,s}$ to 1). Similarly, we have $\sigma(\tau(j)) = j$ for every j. Therefore σ and τ are inverse permutations, and $k = k'$.

Applying Lemma 36, p. 28, to (1), we get $t_i \geq t'_{\sigma(i)}$, and similarly $t'_j \geq t_{\tau(j)}$, so $t_i \geq t'_{\sigma(i)} \geq t_i$. Therefore $(t_1, \ldots, t_k) = \left(t'_{\sigma(1)}, \ldots, t'_{\sigma(k)}\right)$, and Lemma 18′, p. 24, finally yields $s = s'$. **q.e.d.**

This result agrees with [Robinson 82, exercise 6.3.5., p. 177].

PROPOSITION 40: Let $U \leq G_{t,s}$ be a subgroup with $\operatorname{ir}(U) < \infty$. Then U is isomorphic to some $G_{t',s'}$.

Proof: By Kurosh's Theorem 38, p. 29, and Lemma 36, p. 28, U is the free product of some free group and some Δ_h's. Since $\mathrm{ir}\,(U) < \infty$, the number of factors in this product and the rank of the free part are finite. Therefore U is isomorphic to some $G_{t',s'}$. **q.e.d.**

COROLLARY 41: Let U be a free factor of $G_{t,s}$. Then U is isomorphic to some $G_{t',s'}$.

Proof: From the surjective homomorphism $G_{t,s} = U * V \twoheadrightarrow U$ we get $\mathrm{ir}\,(U) \leq \mathrm{ir}\,(G_{t,s}) < \infty$. Now apply Proposition 40, p. 29. **q.e.d.**

DEFINITION 42: A **partition** of r is a tuple $(t_1, \ldots, t_k)$ of positive integers with $t_1 + \cdots + t_k = r$, modulo permutations, i.e. we consider $(t_1, \ldots, t_k)$ and $\left(t_{\sigma(1)}, \ldots, t_{\sigma(k)}\right)$ to be the same partition $\forall \sigma \in S_k$.

DEFINITION 43: A partition $(t'_1, \ldots, t'_{k'})$ is called **finer** than $(t_1, \ldots, t_k)$, in symbols $(t'_1, \ldots, t'_{k'}) \succeq (t_1, \ldots, t_k)$, if inserting parentheses into $r = t'_1 + \cdots + t'_{k'}$ and evaluating the terms within the parentheses produces the display $r = t_1 + \cdots + t_k$. Of course, this implies $k' \geq k$.

This defines a partial ordering on the set of partitions of r.
Example: $(1,1,1,1,1) \succeq (1,1,2,1) \succeq (1,3,1) \succeq (4,1) \succeq (5)$.

THEOREM 44: Given $G_{t,s}$ and $G_{t',s'}$ with $g = g'$, there is a surjective homomorphism $G_{t,s} \twoheadrightarrow G_{t',s'}$ if and only if $s' \geq s$ and $\big(t'_1, \ldots, t'_{k'}, \underbrace{1, \ldots, 1}_{s'}\big) \succeq \big(t_1, \ldots, t_k, \underbrace{1, \ldots, 1}_{s}\big)$.

Proof: "$\Leftarrow$": Repeated use of the homomorphisms $\Delta_{i+j} \twoheadrightarrow \Delta_i * \Delta_j$ and $\Delta_1 \twoheadrightarrow \mathbb{Z}$. $t_1 = t'_1 + \cdots + t'_{i_1}$ gives $\Delta_{t_1} \twoheadrightarrow \Delta_{t'_1} * \cdots * \Delta_{t'_{i_1}}$, and so on. This yields $G_{t,s} = G_{t,0} * \mathbb{Z}^{*s} \twoheadrightarrow G_{t',s'-s} * \mathbb{Z}^{*s} = G_{t',s'}$.
"$\Rightarrow$": Let $\varphi : G_{t,s} \to G_{t',s'}$ be a surjective homomorphism. Write $G_{t,s} = G_1 * \cdots * G_m$ with $m := k+s$, $G_i \cong \Delta_{t_i}$ for $i \leq k$ and $G_i \cong \mathbb{Z}$ for $i > k$. For every $i \in \{1, \ldots, m\}$, consider $\varphi(G_i)$. By Proposition 40, p. 29, and since $\mathrm{ir}\,(\varphi(G_i)) \leq \mathrm{ir}\,(G_i) = \left\{\begin{array}{ll} t_i & \text{for } i \leq k \\ 1 & \text{for } i > k \end{array}\right\} < \infty$, $\varphi(G_i)$ is isomorphic to some $G_{t^{(i)},s^{(i)}}$. Define $r^{(i)}$, $s^{(i)}$ and $g^{(i)} = r^{(i)} + s^{(i)}$ as usual.
Let $G_{T,S}$ be the (abstract) free product $G_{T,S} = G_{t^{(1)},s^{(1)}} * \cdots * G_{t^{(m)},s^{(m)}}$, with $T =$ concatenation of the $t^{(i)}$ and $S = \sum_{i=1}^{m} s^{(i)}$. We have a surjective homomorphism

$$G_{T,S} \cong \varphi(G_1) * \cdots * \varphi(G_m) \longrightarrow\!\!\!\!\!\rightarrow \langle \varphi(G_1), \ldots, \varphi(G_m) \rangle = \varphi(G_{t,s}) = G_{t',s'} \qquad (3)$$

CLAIM 1: $\mathrm{ir}\,(G_i) = g^{(i)}$.
Proof: By (3),

$$g = \mathrm{ir}\,(G_{t,s}) = \sum_{i=1}^{m} \mathrm{ir}\,(G_i) \geq \sum_{i=1}^{m} \mathrm{ir}\,(\varphi(G_i)) = \mathrm{ir}\,(G_{T,S}) \geq \mathrm{ir}\,(G_{t',s'}) = g' = g$$

Therefore, equality must hold for all i: $\mathrm{ir}\,(G_i) = \mathrm{ir}\,(\varphi(G_i)) = g^{(i)}$.
In particular, $\mathrm{ir}\,(G_{T,S}) = g$.
CLAIM 2: $S = s'$.

Proof: Consider the homomorphism $G_{t',s'} \twoheadrightarrow \mathbb{Z}^{*s'}$ which maps all the $\Delta_{t'_j}$ to 1. It kills all the Δ_h-factors of the $G_{t^{(i)},s^{(i)}}$ (we know this by the application of Kurosh's theorem above), therefore induces a surjective homomorphism $\mathbb{Z}^{*s^{(1)}} * \cdots * \mathbb{Z}^{*s^{(m)}} \twoheadrightarrow \left\langle \mathbb{Z}^{*s^{(1)}}, \ldots, \mathbb{Z}^{*s^{(m)}} \right\rangle \twoheadrightarrow \mathbb{Z}^{*s'}$, so $S = \sum_{i=1}^{m} s^{(i)} \geq s'$. On the other hand, applying corollary 35(b), p. 28, to $G_{T,S} \twoheadrightarrow G_{t',s'}$, we get $S \leq s'$.
We know that every $\Delta_{t_\iota^{(i)}}$-factor of a $G_{t^{(i)},s^{(i)}}$ is conjugate to some subgroup of finite index in some $\Delta_{t'_j}$ ($j \in \{1, \ldots, k'\}$). This j is uniquely determined (since the factors of $G_{t',s'}$ are not conjugate to each other). Consider this j as a function of i and ι.
Claim 3: Every $j \in \{1, \ldots, k'\}$ occurs at least once.
Proof: If some $\Delta_{t'_j}$ does not occur, consider the homomorphism $G_{t',s'} \twoheadrightarrow \Delta_{t'_j} * \mathbb{Z}^{*s'}$ which maps all other Δ-factors of $G_{t',s'}$ to 1. Similarly to claim 2, it kills all Δ-factors of the $G_{t^{(i)},s^{(i)}}$, so we get a surjective homomorphism $\mathbb{Z}^{*s^{(1)}} * \cdots * \mathbb{Z}^{*s^{(m)}} \twoheadrightarrow \left\langle \mathbb{Z}^{*s^{(1)}}, \ldots, \mathbb{Z}^{*s^{(m)}} \right\rangle \twoheadrightarrow \Delta_{t'_j} * \mathbb{Z}^{*s'}$. Abelianizing, we see $S = \sum_{i=1}^{m} s^{(i)} \geq 2t'_j + s'$, which contradicts claim 2.
Claim 4: Every $j \in \{1, \ldots, k'\}$ occurs exactly once.
Proof: By claim 3, $\sum_{i=1}^{m} k^{(i)} \geq k'$. Applying corollary 35(c), p. 28, to $G_{T,S} \twoheadrightarrow G_{t',s'}$, we get $\sum_{i=1}^{m} k^{(i)} \leq k'$. Therefore $\sum_{i=1}^{m} k^{(i)} = k'$, and no j can occur more than once.
For each Δ-factor of $G_{T,S}$ (namely $\Delta_{t_\iota^{(i)}}$ with $i \in \{1, \ldots, k\}$, $\iota \in \{1, \ldots, k^{(i)}\}$) and the corresponding $j \in \{1, \ldots, k'\}$ we know by Lemma 36, p. 28, that $t_\iota^{(i)} \geq t'_j$. Summing up, we have

$$r' = g' - s' = \operatorname{ir}(G_{T,S}) - S = \sum_i r^{(i)} = \sum_{i,\iota} t_\iota^{(i)} \geq \sum_j t'_j = r'$$

We must therefore have $t_\iota^{(i)} = t'_j$ for every i, ι. So the index in $\Delta_{t'_j}$ of the subgroup conjugate to $\Delta_{t_\iota^{(i)}}$ is 1.
Summarizing, every $\Delta_{t_\iota^{(i)}}$-factor of $G_{t^{(i)},s^{(i)}}$ is conjugate to a $\Delta_{t'_j}$, with $j \in \{1, \ldots, k'\}$ uniquely determined by i and ι, and every $j \in \{1, \ldots, k'\}$ occurs exactly once. In particular, $T = t'$ (up to permutation).
We also have for $i \in \{1, \ldots, m\}$:

$$\left\{ \begin{matrix} t_i & \text{for } i \leq k \\ 1 & \text{for } i > k \end{matrix} \right\} = \operatorname{ir}(G_i) = g^{(i)} = r^{(i)} + s^{(i)} = \sum_\iota t_\iota^{(i)} + \underbrace{1 + \cdots + 1}_{s^{(i)}}$$

This and claim 2 prove that $(t'_1, \ldots, t'_{k'}, 1, \ldots, 1) \succeq (t_1, \ldots, t_k, 1, \ldots, 1)$. The inequality $s' \geq s$ was already proved in corollary 35(b), p. 28. **q.e.d.**
Note that in the preceding proof, $T = t'$ and $S = s'$. By lemma 25, p. 25, the surjective homomorphism (3) is an isomorphism. This means that $\varphi(G_{t,s})$ is the *free* product of $\varphi(G_1), \ldots, \varphi(G_m)$.

Corollary 45: Let $G_{t,s}$ and $G_{t',s'}$ with $g = g'$ be given, and $\varphi : G_{t,s} \twoheadrightarrow G_{t',s'}$ a surjective homomorphism. If $G_{t,s} = G_1 * \cdots * G_m$ is a free decomposition of $G_{t,s}$, then $G_{t',s'} = \varphi(G_1) * \cdots * \varphi(G_m)$ is a free decomposition of $G_{t',s'}$.

Proof: We just saw that this holds for every maximal free decomposition ($m = k + s$) of $G_{t,s}$. In general, G_i is isomorphic to some $G_{t^{(i)},s^{(i)}}$ by Corollary 41, p. 30. Let $G_i = \mathop{\ast}\limits_{\iota=1}^{m^{(i)}} G_\iota^{(i)}$ be the maximal free decomposition of G_i, so

$$G_{t,s} = \mathop{\ast}_{\substack{i\in\{1,\ldots,m\}\\ \iota\in\left\{1,\ldots,m^{(i)}\right\}}} G_\iota^{(i)}$$

is a maximal free decomposition of $G_{t,s}$. We thus have

$$G_{t',s'} = \mathop{\ast}_{\substack{i\in\{1,\ldots,m\}\\ \iota\in\left\{1,\ldots,m^{(i)}\right\}}} \varphi\left(G_\iota^{(i)}\right)$$

so $\varphi(G_i) = \left\langle \varphi\left(G_1^{(i)}\right), \ldots, \varphi\left(G_{m^{(i)}}^{(i)}\right) \right\rangle = \mathop{\ast}\limits_{\iota\in\left\{1,\ldots,m^{(i)}\right\}} \varphi\left(G_\iota^{(i)}\right)$, and

$$G_{t',s'} = \mathop{\ast}_{i\in\{1,\ldots,m\}} \varphi(G_i)$$

q.e.d.

2. The coverings

2.1. Coverings.

In this section we summarize the definitions and results of the theory of coverings. It follows from the treatment in [Forster 77, §§ 4-5] and [Cohen 89, chapter 4.3 and chapter 7] that coverings can be defined within the framework of topological spaces: manifolds are not required.
Let $\mathcal{C}$ be one of the categories

$\mathcal{C} = \mathbf{Top}$ (topological spaces),
$\mathcal{C} = \mathbf{HdTop}$ (Hausdorff spaces),
$\mathcal{C} = \mathbf{slscHdTop}$ (semi-locally simply connected Hausdorff spaces, see [Cohen 89, chapter 6, p. 154])
$\mathcal{C} = \mathbf{C}^n$ manifolds ($n \in \mathbb{N}_0 \cup \{\infty\}$),
$\mathcal{C} = \mathbf{RS}$,
$\mathcal{C} = \mathbf{RSN}$.

(Essentially, a category whose objects are topological spaces and whose morphisms are continuous mappings with some local conditions.)

$$\begin{tikzcd}
 & \mathbf{Top} \arrow[d, no head] & \\
 & \mathbf{HdTop} \arrow[d, no head] & \\
 & \mathbf{slscHdTop} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbf{RSN} \arrow[dr, no head] & & \mathbf{C}^\infty - \mathbf{mf} \arrow[dl, no head] \\
 & \mathbf{RS} &
\end{tikzcd}$$

Definition 46: Let $X \in \mathrm{Ob}\,\mathcal{C}$. An **unramified covering (UC)** of X is a $Y \in \mathrm{Ob}\,\mathcal{C}$ together with a surjective and locally homeomorphic morphism $p \in \mathrm{Mor}_{\mathcal{C}}\,(Y, X)$ (called the **projection**). An **unlimited, unramified covering (UUC)** is an unramified covering which has the path lifting property: Given any curve γ in X and any $y_0 \in p^{-1}(\gamma(0))$, there is a curve δ in Y with $p(\delta(t)) = \gamma(t)\ \forall t \in [0,1]$ – we say that δ lies **over** γ – and $\delta(0) = y_0$.

See [Springer 57, Chap. 4-1] and [Forster 77, 4.14, 4.19].
Note that in $\mathcal{C} = \mathbf{RSN}$, coverings preserve nodes: $y \in Y$ is a node of $Y \;\Leftrightarrow\; p(y) \in X$ is a node of X.

Definition 47: Two UUCs

$$Y_1 \xrightarrow{p_1} X$$

and

$$Y_2 \xrightarrow{p_2} X$$

are called **isomorphic** if there is a $\mathcal{C}$-isomorphism $f : Y_1 \to Y_2$ such that the diagram

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes.

From now on, assume that $\mathcal{C}$ is a subcategory of **slscHdTop**. (The second part of Proposition 48, p. 34, needs this, see [Cohen 89, chapter 6, p. 153-155].) Fortunately RSNs and manifolds are locally contractible, hence locally simply connected, hence semi-locally simply connected.

Proposition 48: To every isomorphism class of UUCs Y of X there corresponds a conjugacy class of subgroups U of $\pi_1(X)$:

- To a UUC Y, one associates

$$U := p(\pi_1(Y)) = \{p(\gamma) : \gamma \text{ a closed path in } Y\}$$

- Conversely, given $U \leq \pi_1(X, x_0)$, Y can be constructed as follows: As a point set, $Y := \{\gamma : \gamma \text{ path in } X \text{ beginning at } x_0\}/_{\sim}$, where $\gamma_1 \sim \gamma_2 \;:\Leftrightarrow\; \gamma_1$ and γ_2 have the same endpoint and $\gamma_1 {\gamma_2}^{-1} \in U$. Then lift the topology and the $\mathcal{C}$-structure from X to Y (as in [Forster 77, 4.6.]).

(See [Springer 57, Chap. 4-3, Theorem 4-6, p. 86].)
The trivial covering $Y = X$ corresponds to $U = \pi_1(X)$, and the **universal covering** $Y = \tilde{X}$ corresponds to $U = \{1\}$.

Definition 49: For a UUC $Y \xrightarrow{p} X$, the group of **covering transformations** (german: **Decktransformationen**) is defined as

$$\mathcal{D}eck(Y/X) := \{\alpha \in \mathrm{Aut}_{\mathcal{C}}(Y) : p \circ \alpha = p\} \cong {N_{\pi_1(X)}(U)}/_{U}$$

(See [Cohen 89, Prop. 13, p. 164] and also [Ahlfors, Sario 60, I.19A].)

Definition 50: A covering is called **regular** (or **normal** or **galois**) if $U \trianglelefteq \pi_1(X)$. In this case, $X \cong {Y}/_{\mathcal{D}eck(Y/X)}$ and $\mathcal{D}eck(Y/X) \cong {\pi_1(X)}/_{U}$.

Corollary 51: $\mathcal{D}eck\left(\tilde{X}/X\right) \cong \pi_1(X)$.

Proposition 52: For every UUC $Y \xrightarrow{p} X$, there is a UUC $\tilde{X} \xrightarrow{f} Y$ such that the diagram

$$\begin{tikzcd}
\tilde{X} \arrow[dd] \arrow[dr, "f"] & \\
& Y \arrow[dl, "p"] \\
X &
\end{tikzcd}$$

commutes, and there is a subgroup $U \leq \mathcal{D}eck\left(\tilde{X}/X\right)$ such that $Y = {}^{\tilde{X}}/_{U}$. Since $\tilde{X}$ is simply connected, $U = \mathcal{D}eck\left(\tilde{X}/Y\right) \cong \pi_1(Y)$.

Proof: [Forster 77, 5.9.]

Corollary 53: If $Y \xrightarrow{p} X$ is a regular UUC,

$$\mathcal{D}eck(Y/X) \cong {}^{\pi_1(X)}/_{\pi_1(Y)}$$

The fact $\pi_1(Y) \hookrightarrow \pi_1(X)$ is also proved in [Cohen 89, Theorem 2, p. 153].

Proposition 54: Let $Y \in \mathrm{Ob}\,\mathcal{C}$, and let some $G \leq \mathrm{Aut}_{\mathcal{C}}(Y)$ operate freely discontinuously on Y. Then $X := {}^{Y}/_{G} \in \mathrm{Ob}\,\mathcal{C}$, and $Y \longrightarrow X$ is a regular covering, with $\mathcal{D}eck(Y/X) = G$.

Proof: [Cohen 89], [Farkas, Kra 80], [Forster 77], [Springer 57], [Stillwell 80], see also [Stöcker, Zieschang 94, Satz 6.1.7.].

2.2. Planarity.

Definition 55:

(a) An orientable surface X is called **topologically planar** if it is homeomorphic to an open connected subset of $\widehat{\mathbb{C}}$. Or, equivalently, if every simple closed curve on X is dividing. (See [Maskit 88, X.A.2, p. 250].)
(b) A RS X is called **conformally planar** if it is conformally equivalent to an open connected subset of $\widehat{\mathbb{C}}$.

By [Ahlfors, Sario 60, II.12C, p. 144] and [Ahlfors, Sario 60, I.46A, p. 107], a RS X is a "polyhedron", i.e. an orientable surface. By [Ahlfors, Sario 60, I.44C, p.104], if X is homologically planar, it is also topologically planar. Furthermore we have:

Proposition 56: Let X be a RS. Then: X is topologically planar $\Leftrightarrow$ X is conformally planar.

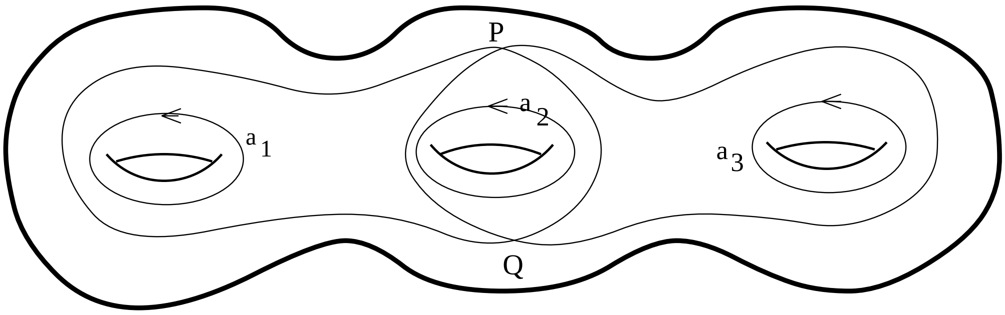
Proof: " $\Rightarrow$ " [Ahlfors, Sario 60, IV.11C, p. 177]. **q.e.d.**
In this case, we call X **planar**.

Note 57: A regular covering of a planar surface need *not* be planar.

Proof: We have the following counterexample: Let $g \geq 3$, X_g a compact RS of genus g, $N_X := \langle a_1, \ldots, a_g \rangle_{\Delta_g}$, $N_Y := \langle a_1 a_2, a_2 a_3 \rangle_{\Delta_g}$ and X, Y the regular coverings of X_g corresponding to N_X and N_Y, respectively. Y is a regular covering of X, since $N_Y \trianglelefteq \pi_1(X_g)$, $N_Y \leq N_X \;\Rightarrow\; N_Y \trianglelefteq N_X = \pi_1(X)$.

$$\begin{array}{c} Y \\ \Big| \\ X \\ \Big| \\ X_g \end{array}$$

X is a Schottky covering of X_g, therefore planar (by a class O_{AD} argument or directly by the planarity theorem [Maskit 88, X.A.3, p. 250]).
Claim: Y is not planar.
The curves γ_1, γ_2 on X_g



with $[\gamma_1] = a_1 a_2$ and $[\gamma_2] = a_2 a_3$ (considered as beginning and ending in P) lift to closed loops δ_1, δ_2 on Y, by definition of N_Y.
Claim: The intersection number of δ_1 and δ_2 is ± 1.
δ_1 and δ_2 intersect transversally over P, since γ_1 and γ_2 intersect transversally at P. But δ_1 and δ_2 have no other point in common. Because if they had some point in common over Q, a_2 would lift to a closed loop in Y, i.e. $a_2 \in N_Y$.
Claim: $a_2 \notin N_Y$.
Consider the homomorphism

$$\varphi : \left\{ \begin{array}{ccc} \Delta_g & \to & \left(\mathbb{Z}/2\mathbb{Z}, +\right) \\ a_1, \ldots, a_g & \mapsto & 1 \\ b_1, \ldots, b_g & \mapsto & 0 \end{array} \right\}$$

Obviously $N_Y \leq \operatorname{kern} \varphi$ but $a_2 \notin \operatorname{kern} \varphi$. **q.e.d.**
We will now investigate a sufficient condition for Y to be planar.

LEMMA 58: Let X be a CRSN of genus g such that the components X_j of X are RS (i.e. $N_{X_j} = \emptyset$ for every component X_j) and the intersection graph G_X is a tree (i.e. $\chi(G_X) = 0$). Let $\varphi : \pi_1(X) \twoheadrightarrow G_{t,s}$ be a surjective homomorphism, with $\mathrm{ir}\,(\pi_1(X)) = g = \mathrm{ir}\,(G_{t,s})$. Then $\mathrm{kern}\,\varphi = \langle [\gamma_1], \ldots, [\gamma_m] \rangle_{\pi_1(X)}$ with some *disjoint* simple closed curves $\gamma_1, \ldots, \gamma_m$ in $X \setminus N_X$.

Proof: Put $N := \mathrm{kern}\,\varphi$. Let $g^{[2]}(X) := \sum_{X_j} g(X_j)^2 \in \mathbb{N}_0$. We prove the result by induction on $g^{[2]}(X)$.
If $g^{[2]}(X) = 0$ or $N = \{1\}$, the result is trivial: choose $m = 0$.
So let us assume $N \neq \{1\}$. We have $\varphi : \pi_1(X) = \mathop{\ast}_{X_j} \pi_1(X_j) \to G_{t,s}$. For every component X_j of X, let $\varphi_j := \varphi|_{\pi_1(X_j)} : \pi_1(X_j) \to \varphi(\pi_1(X_j)) \leq G_{t,s}$. By Corollary 45, p. 32, $G_{t,s} = \mathop{\ast}_{X_j} \varphi(\pi_1(X_j))$, and if every φ_j were an isomorphism, then φ would be an isomorphism as well. So there is some non-injective φ_j, i.e. $N \cap \pi_1(X_j) = \mathrm{kern}\,\varphi_j \neq \{1\}$. Choose a $[\gamma] \in (N \cap \pi_1(X_j)) \setminus \{1\}$.
W.l.o.g. γ can be assumed to be a simple closed curve. (To see this: Since X_j is a compact surface, the homotopy class $[\gamma]$ contains a geodesic. Assume γ to be this geodesic. So γ is the power of a simple closed geodesic δ: $\gamma = \delta^n$, $n \in \mathbb{N}$. Since $G_{t,s}$ is torsion-free (Note 29, p. 26), $\varphi([\delta])^n = \varphi([\gamma]) = 1$ implies $\varphi([\delta]) = 1$. Now replace $[\gamma]$ by $[\delta] \in (N \cap \pi_1(X_j)) \setminus \{1\}$.)
W.l.o.g. γ can be assumed to lie in $X_j \setminus N_X$. (One can avoid the finitely many points $\in N_X \cap X_j$ by choosing another simple closed curve in the same homotopy class.)
Now, φ factors through ${\pi_1(X)}/{\langle\gamma\rangle_{\pi_1(X)}}$:

$$\varphi = \left(\pi_1(X) \twoheadrightarrow {\pi_1(X)}/{\langle\gamma\rangle_{\pi_1(X)}} \xrightarrow{\psi} G_{t,s}\right)$$

Case 1: γ divides X_j into two components U_1 and U_2, i.e. U_1, U_2 open, $U_1 \dot\cup U_2 = X_j \setminus \gamma$, $\overline{U_i} = U_i \cup \gamma$. By the Seifert – van Kampen theorem ([Stillwell 80, Chapter 3.4], [Stöcker, Zieschang 94, Satz 5.3.11.], [Cohen 89, Corollary 2, p. 86]), $\pi_1(X_j) = \pi_1\left(\overline{U_1}\right) \mathop{\ast}_{\langle\gamma\rangle} \pi_1\left(\overline{U_2}\right)$, so

$$ {\pi_1(X_j)}/{\langle\gamma\rangle_{\pi_1(X_j)}} = {\pi_1\left(\overline{U_1}\right)}/{\langle\gamma\rangle_{\pi_1\left(\overline{U_1}\right)}} \ast {\pi_1\left(\overline{U_2}\right)}/{\langle\gamma\rangle_{\pi_1\left(\overline{U_2}\right)}} $$

Let Y be the RSN which arises from X by "pinching" γ, i.e. in Y X_j is replaced by $U_1 \dot\cup \{P_0\} \dot\cup U_2$, the union of two RS $Z_1 := U_1 \cup \{P_0\}$ and $Z_2 := \{P_0\} \cup U_2$. From ${\pi_1(X_j)}/{\langle\gamma\rangle_{\pi_1(X_j)}} = \pi_1(Z_1) \ast \pi_1(Z_2)$ we get

$$ {\pi_1(X)}/{\langle\gamma\rangle_{\pi_1(X)}} = \pi_1(Y) $$

We now apply the induction hypothesis to $\psi : \pi_1(Y) \twoheadrightarrow G_{t,s}$. (This is possible: $g(X_j) = g(Z_1) + g(Z_2)$, $g(Z_1) > 0$, $g(Z_2) > 0$. (If $g(Z_i) = 0$, γ would be homotopic in $U_i \cup \gamma$ to a constant curve, so $[\gamma] = 1$.) $g^{[2]}(X) - g^{[2]}(Y) = g(X_j)^2 - \Big(g(Z_1)^2 + g(Z_2)^2\Big) = 2 \cdot g(Z_1) \cdot g(Z_2) > 0$.)

The induction hypothesis gives $\operatorname{kern} \psi = \langle [\gamma_1], \ldots, [\gamma_m] \rangle_{\trianglelefteq}$ with some disjoint simple closed curves $\gamma_1, \ldots, \gamma_m$. So $\operatorname{kern} \varphi = \langle \operatorname{kern} \psi, [\gamma] \rangle_{\pi_1(X)} = \langle [\gamma_1], \ldots, [\gamma_m], [\gamma] \rangle_{\pi_1(X)}$. γ is disjoint from $\gamma_1, \ldots, \gamma_m$ (in X) because $\gamma_1, \ldots, \gamma_m$ do not pass through P_0 (in Y).

Case 2: γ is a non-separating curve on X_j. Put $g_j := g(X_j)$. By [Birman 77, Lemma 4.5, p. 181], the homeomorphism group of a CRS operates transitively on the set of non-separating curves. We may therefore assume that $\gamma = a_1$, the first of the curves making up the standard presentation $\pi_1(X_j) = \big\langle a_1, b_1, \ldots, a_{g_j}, b_{g_j} \mid \prod_{i=1}^{g_j} [a_i, b_i] = 1 \big\rangle$ of $\pi_1(X_j)$. Let $\delta := b_1$, a curve which intersects γ in one point only. We now have

$$\pi_1(X_j) / \langle \gamma \rangle_{\pi_1(X_j)} = \langle \delta \rangle * \Delta_{g_j - 1}$$

γ and δ can be viewed as the curves passing "along" and "beneath" the first handle of X_j, respectively. $\Delta_{g_j-1} = \pi_1(Z)$, where Z arises from X_j by pinching γ and blowing up the resulting singularity (or, equivalently, by cutting X_j along γ and filling the resulting two open ends with two disks). Let Y be the RSN X with X_j replaced by Z. We have

$$\psi : \pi_1(Y) * \langle \delta \rangle \twoheadrightarrow G_{t,s}$$

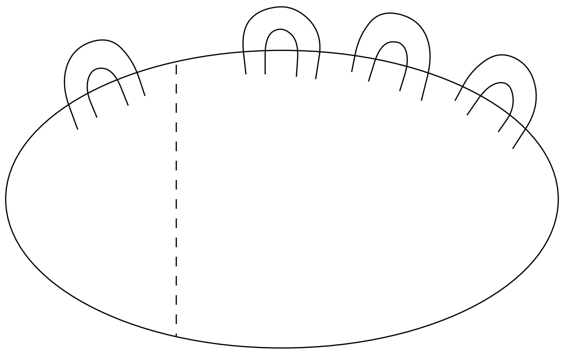
By Corollary 45, p. 32, $G_{t,s} = \psi(\pi_1(Y)) * \psi(\langle \delta \rangle)$. From $g = \operatorname{ir}(G_{t,s}) = \operatorname{ir}(\psi(\pi_1(Y))) + \operatorname{ir}(\psi(\langle \delta \rangle)) \leq \operatorname{ir}(\pi_1(Y)) + \operatorname{ir}(\langle \delta \rangle) = (g-1) + 1$ we infer that $\operatorname{ir}(\psi(\langle \delta \rangle)) = 1$, so $\psi(\langle \delta \rangle) \cong \mathbb{Z}$ and $\psi(\pi_1(Y)) \cong G_{t,s-1}$.

We now apply the induction hypothesis to $\psi|_{\pi_1(Y)} : \pi_1(Y) \twoheadrightarrow \psi(\pi_1(Y)) \cong G_{t,s-1}$. (This is possible since $g^{[2]}(X) - g^{[2]}(Y) = g(X_j)^2 - g(Z)^2 = {g_j}^2 - (g_j - 1)^2 > 0$.) The induction hypothesis gives

$$\begin{aligned}
&\operatorname{kern}\Big(\psi|_{\pi_1(Y)}\Big) = \langle [\gamma_1], \ldots, [\gamma_m] \rangle_{\pi_1(Y)} \\
\Rightarrow\ &\operatorname{kern} \psi = \langle [\gamma_1], \ldots, [\gamma_m] \rangle_{\pi_1(Y) * \langle \delta \rangle} \\
\Rightarrow\ &\operatorname{kern} \varphi = \langle \operatorname{kern} \psi, [\gamma] \rangle_{\pi_1(X)} = \langle [\gamma_1], \ldots, [\gamma_m], [\gamma] \rangle_{\pi_1(X)}
\end{aligned}$$

Here we transport those curves among the $\gamma_1, \ldots, \gamma_m$ which lie in Z to simple closed curves in X_j. They can be chosen to be disjoint from γ (and δ as well) since

$\langle \gamma, \delta \rangle$ and $\langle a_2, b_2, \ldots, a_{g_j}, b_{g_j} \rangle = \pi_1(Z)$ "live" in disjoint parts of X_j:



$\langle \gamma, \delta \rangle$ $\langle a_2, b_2, \ldots, a_{g_j}, b_{g_j} \rangle$

q.e.d.

<u>THEOREM 59:</u> Let X be a CRS of genus g, and $Y \longrightarrow X$ a regular covering of RS. Then the following are equivalent:

(i) Y planar and $\mathcal{D}eck(Y/X)$ torsion-free,

(ii) $\operatorname{kern}(\pi_1(X) \to \mathcal{D}eck(Y/X)) = \langle [\gamma_1], \ldots, [\gamma_m] \rangle_{\pi_1(X)}$ where $\gamma_1, \ldots, \gamma_m$ are disjoint simple closed curves on X.

(iii) $\mathcal{D}eck(Y/X)$ isomorphic to some $G_{t,s}$, and $\operatorname{ir}(\mathcal{D}eck(Y/X)) = g$.

Note that we always have $\operatorname{ir}(\mathcal{D}eck(Y/X)) = \operatorname{ir}\left({}^{\pi_1(X)}/_{\pi_1(Y)} \right) \leq \operatorname{ir}(\pi_1(X)) = g$, by Lemma 14, p. 20.

Proof of the theorem: Let $\varphi : \pi_1(X) \twoheadrightarrow {}^{\pi_1(X)}/_{\pi_1(Y)} \cong \mathcal{D}eck(Y/X)$ be the canonical homomorphism and $N := \operatorname{kern}\varphi$.

"(i) $\Rightarrow$ (ii)": By the planarity theorem [Maskit 88, X.A.4, p. 250-251], $N = \langle [\gamma_1]^{n_1}, \ldots, [\gamma_m]^{n_m} \rangle_{\pi_1(X)}$ is generated by powers of disjoint simple closed curves $\gamma_1, \ldots, \gamma_m$. Since $\mathcal{D}eck(Y/X)$ is torsion-free, $\varphi([\gamma_i])^{n_i} = \varphi([\gamma_i]^{n_i}) = 1$ implies $\varphi([\gamma_i]) = 1$, so $[\gamma_i] \in N$. Therefore $N = \langle [\gamma_1], \ldots, [\gamma_m] \rangle_{\pi_1(X)}$.

"(ii) $\Rightarrow$ (iii)": Let Z be the RSN one obtains from X by pinching $\gamma_1, \ldots, \gamma_m$. (In detail: As a point set, $Z := (X \setminus (\gamma_1 \cup \cdots \cup \gamma_m)) \dot\cup \{z_1, \ldots, z_m\}$. Take the final topology that makes the map

$$\left\{ \begin{array}{r} X \to Z \\ x \in X \setminus (\gamma_1 \cup \cdots \cup \gamma_m) \mapsto x \\ x \in \gamma_i \mapsto z_i \end{array} \right\}$$

continuous. Every z_i has two neighbourhood disks, corresponding to the two neighbourhood strips of γ_i. (Recall that X is an oriented surface.))

Now $\mathcal{D}eck(Y/X) \cong {}^{\pi_1(X)}/_N \cong \pi_1(Z)$, which is isomorphic to some $G_{t,s}$ by Theorem 11, p. 17. Furthermore, pinching doesn't change the genus. (This is seen by induction on m, distiguishing the case of separating and non-separating curves again.) So $\operatorname{ir}(\mathcal{D}eck(Y/X)) = \operatorname{ir}(\pi_1(Z)) = g(Z) = g(X) = g$.

"(iii) $\Rightarrow$ (ii)": Apply Lemma 58, p. 37, to X and φ.
"(ii) $\Rightarrow$ (i)": By the planarity theorem [Maskit 88, X.A.3, p. 250], Y is planar. We have already proved that (ii) implies (iii), so $\mathcal{D}eck(Y/X)$ is some $G_{t,s}$, hence torsion-free by Note 29, p. 26. **q.e.d.**

<u>DEFINITION 60:</u> For a group G, define

$$G^{\mathrm{notor}} := {}^{G}\big/_{\langle\{x \in G : x \text{ of finite order in } G\}\rangle}$$

This is a factor group of G with "less" torsion than G. If G is abelian, G^{notor} is torsion-free.

<u>LEMMA 61:</u>

(a) If $\varphi : G \to H$ is a group homomorphism and H is torsion-free, then φ factors through G^{notor}:

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr] & & H \\ & G^{\mathrm{notor}} \arrow[ur] & \end{tikzcd}$$

(b) $\mathrm{ir}\,(G^{\mathrm{notor}}) = \mathrm{ir}\,(G)$.

Proof:

(a) If $x \in G$ is of finite order, say $x^n = 1$, then $\varphi(x)^n = \varphi(x^n) = \varphi(1) = 1$, and since H is torsion-free, $\varphi(x) = 1$. So $\mathrm{kern}\,\varphi \geq \langle\{x \in G : x \text{ of finite order in } G\}\rangle$.

(b) Apply (a) to $H = \mathbb{Z}^{*n}$.

q.e.d.

<u>COROLLARY 62:</u> Let X be a CRS of genus g, and $Y \longrightarrow X$ a regular covering of RS. If Y is planar, then $\mathcal{D}eck(Y/X)^{\mathrm{notor}}$ is isomorphic to some $G_{t,s}$, and $\mathrm{ir}\,(\mathcal{D}eck(Y/X)) = g$.

Proof: Consider $\varphi : \pi_1(X) \twoheadrightarrow \mathcal{D}eck(Y/X)$ and

$$\psi : \big(\pi_1(X) \xrightarrow{\ \varphi\ } \!\!\!\!\rightarrow \mathcal{D}eck(Y/X) \longrightarrow\!\!\!\!\rightarrow \mathcal{D}eck(Y/X)^{\mathrm{notor}}\big)$$

By the planarity theorem [Maskit 88, X.A.4, p. 250-251], $\mathrm{kern}\,\varphi = \langle[\gamma_1]^{n_1}, \ldots, [\gamma_m]^{n_m}\rangle_{\pi_1(X)}$ is generated by powers of disjoint simple closed curves $\gamma_1, \ldots, \gamma_m$. Since $\varphi([\gamma_i])$ is of finite order $\leq n_i$ in $\mathcal{D}eck(Y/X)$, $\psi([\gamma_i]) = 1$. So $\mathrm{kern}\,\psi \geq N := \langle[\gamma_1], \ldots, [\gamma_m]\rangle_{\pi_1(X)} \geq \mathrm{kern}\,\varphi$. We have the commutative diagram

$$\begin{tikzcd} & & {}^{\pi_1(X)}\big/_{N} \arrow[dd] \\ \pi_1(X) \arrow[urr] \arrow[r, "\varphi"] \arrow[drr, "\psi"'] & \mathcal{D}eck(Y/X) \arrow[ur] \arrow[dr] & \\ & & \mathcal{D}eck(Y/X)^{\mathrm{notor}} \end{tikzcd}$$

Since $\pi_1(X)/_N$ is already torsion-free by Theorem 59, p. 39, there is a homomorphism $\mathcal{D}eck(Y/X)^{\text{notor}} \to \pi_1(X)/_N$ in the above diagram, by Lemma 61(a), p. 40.

$$\begin{tikzcd}
 & & \pi_1(X)/_N \arrow[dd, shift right] \\
\pi_1(X) \arrow[urr, two heads] \arrow[r, "\varphi"] \arrow[drr, two heads, "\psi"'] & \mathcal{D}eck(Y/X) \arrow[ur, two heads] \arrow[dr, two heads] & \\
 & & \mathcal{D}eck(Y/X)^{\text{notor}} \arrow[uu, shift right]
\end{tikzcd}$$

Hence $\operatorname{kern}\psi \leq N$. We conclude that $\operatorname{kern}\psi = N$, $\mathcal{D}eck(Y/X)^{\text{notor}} \cong \pi_1(X)/_N$, and by theorem 59, p. 39, this is isomorphic to some $G_{t,s}$ with $\operatorname{ir}(\mathcal{D}eck(Y/X)) = \operatorname{ir}\left(\mathcal{D}eck(Y/X)^{\text{notor}}\right) = \operatorname{ir}\left(\pi_1(X)/_N\right) = g$. **q.e.d.**

<u>NOTE 63:</u> If the kernel of $\varepsilon : \mathcal{D}eck(Y/X) \twoheadrightarrow \mathbb{Z}^{*g}$ is either $= \{1\}$ or $\cong \mathbb{Z}$, then Y is of class O_{AD}. (See [Ahlfors, Sario 60, IV.1B] for the definition and [Ahlfors, Sario 60, IV.2B and IV.2D] for the main properties of class O_{AD}.)

Proof: Use the criterion from [Ahlfors, Sario 60, IV.19A-19E]: Tesselate Y with lifts of the fundamental polygon P of X, i.e. $Y = \bigcup_{g \in \mathcal{D}eck(Y/X)} g^{-1}\tilde{P}$. The polygon $g^{-1}\tilde{P}$ is said to be of **generation** n, if the length $l(g)$ is $= n$. The set of polygons of generation n splits into connected components. Let $\overline{\tau}_n$ be the maximum number of polygons in such a component. We have to prove that $\sum_{n=1}^{\infty} \frac{1}{\overline{\tau}_n} = \infty$. If $\operatorname{kern}\varepsilon = \{1\}$, $\mathcal{D}eck(Y/X) = \mathbb{Z}^{*g}$, and the length of a word $g = g_1{}^{n_1} \cdots g_k{}^{n_k}$ in the free group is $l(g) = |n_1| + \cdots + |n_k| = n$. We get $\overline{\tau}_n = O(|n_k|) = O(n)$ and $\sum_n \frac{1}{\overline{\tau}_n} = \infty$. If $\operatorname{kern}\varepsilon \cong \mathbb{Z}$, write $\mathcal{D}eck(Y/X) = \mathbb{Z}^{*g}{h_0}^{\mathbb{Z}}$. For $g = g_1{}^{n_1} \cdots g_k{}^{n_k} h_0{}^{n_0}$ we have $l(g) = \max(|n_1| + \cdots + |n_k|, |n_0|)$. We get $\overline{\tau}_n = O(|n_k| + |n_0|) = O(n)$ and $\sum_n \frac{1}{\overline{\tau}_n} = \infty$. **q.e.d.**

2.3. $\mathcal{M}_{g,G}$.

DEFINITION 64: The **moduli space** $\mathcal{M}_g$ is the normal complex space of CRS of genus g modulo conformal equivalence. (See [Nag 88, 2.8.2.].)

From now on, let $g \geq 2$.

DEFINITION 65: The **compactified moduli space** $\overline{\mathcal{M}_g}$ is the set of SCRSN of genus g modulo isomorphy in the category RSN. $\overline{\mathcal{M}_g}$ is the Deligne-Mumford compactification of $\mathcal{M}_g$ and known to be a projective variety over $\mathbb{C}$. (See [Nag 88, pp. 301, 405], [Mumford 75], [Gieseker 82], [Knudsen 83].)

DEFINITION 66: For G a group of the form $G = G_{t,s}$, $g = r + s$, (see section 1.3) define

$$\mathcal{M}_{g,G} := \{X \text{ SCRSN of genus } g : \exists \text{ surjective group hom. } \varepsilon : \pi_1(X) \to G\}/_{\cong}$$

Obviously (and by sections 1.5 and 1.7)

$$\mathcal{M}_g = \mathcal{M}_{g,\Delta_g} \subset \mathcal{M}_{g,G} \subset \mathcal{M}_{g,\mathbb{Z}^{*g}} = \overline{\mathcal{M}_g}$$

For $X \in \mathcal{M}_{g,G}$ and $\varepsilon : \pi_1(X) \to G$ surjective, let $U := \operatorname{kern} \varepsilon$. We will consider the covering $Y \xrightarrow{p} X$ in the category **RSN** with $\mathcal{D}eck(Y/X) \cong {}^{\pi_1(X)}/_U \cong G$.

2.4. The G-covering of a RS.

THEOREM 67: There is some $X \in \mathcal{M}_{g,G_{t,s}}$ and $\varepsilon : \pi_1(X) \twoheadrightarrow G$ such that the covering surface Y is planar: $Y \subset \widehat{\mathbb{C}}$, $G_{t,s} \cong G \leq \mathrm{PGL}_2(\mathbb{C})$ acting discontinuously in Y, and $X = {}^Y/_G$.

More precisely, we have:

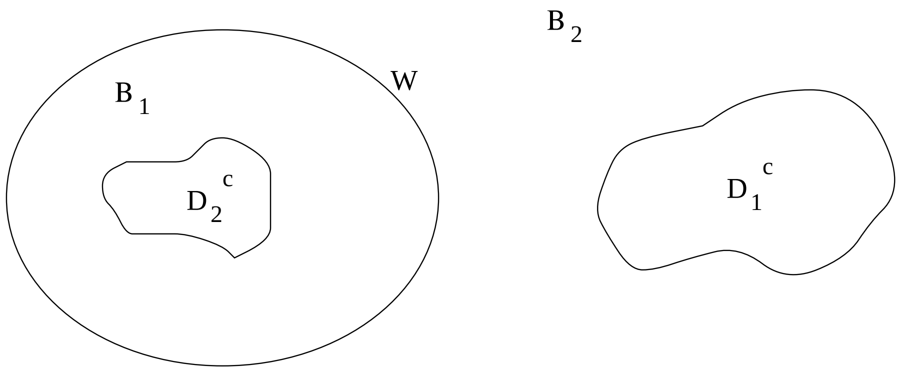
THEOREM 67′: Every $G_{t,s}$ can be realized as covering transformation group of a CRS. There is an embedding $G_{t,s} \cong G \leq \mathrm{PGL}_2(\mathbb{C})$ such that:

(i) The **region of discontinuity** Ω of G (which is, by definition, an open subset of $\widehat{\mathbb{C}}$) is non-empty, i.e. G is a **Kleinian group**.
(ii) G is purely loxodromic, if all $t_j > 1$.
(iii) There is a distinguished G-invariant component Ω' of Ω such that a fundamental domain for ${}^{\Omega'}/_G$ can be constructed as the intersection of the following:
- a hyperbolic $(4t_1)$-gon,
- $\ldots$,
- a hyperbolic $(4t_k)$-gon,

with $2s$ disks removed.
(iv) G is geometrically finite.
(v) $X = {}^{\Omega'}/_G$ is a CRS of genus $g = r + s$.
(vi) $\mathcal{D}eck(\Omega'/X) \cong G \cong G_{t,s}$.

General background about Kleinian groups can be found in [Maskit 88]. Recall that while Ω denotes the region of discontinuity of G, ${}^{o}\Omega$ denotes the region where G operates freely discontinuously, i.e. ${}^{o}\Omega = \Omega \setminus \{\text{fixed points of elliptic elements of } G\}$ ($= \Omega$ if G is torsion-free).
Proof: (i)-(iv) By induction on s.
Case 1: $s = 0$. We use the First Combination Theorem by Klein and Maskit ([Maskit 88], [Maskit 93]). The theorem proved in [Maskit 88, VII.C], specialized with $J = \{\mathrm{id}\}$, reads as follows:

Combination Theorem 1: Let $G_1, G_2 \leq \mathrm{PGL}_2(\mathbb{C})$ be discrete groups and W a simple closed curve dividing $\widehat{\mathbb{C}}$ into B_1 and B_2 such that $g(B_1{}^{o}) \subset B_2{}^{o}$ $\forall g \in G_1 \setminus \{\mathrm{id}\}$, $g(B_2{}^{o}) \subset B_1{}^{o}$ $\forall g \in G_2 \setminus \{\mathrm{id}\}$. Assume B_i is a G_i-**block**, i.e. $B_i \subset \Omega(G_i)$ and B_i contained in some fundamental domain D_i of G_i, $\forall i \in \{1,2\}$, such that $D_1 \cap W = D_2 \cap W$. Also assume $B_1{}^{o} \cap \left(\bigcup\limits_{g \in G_2} g(B_2{}^{o}) \right) \subsetneq B_1{}^{o}$ or $B_2{}^{o} \cap \left(\bigcup\limits_{g \in G_1} g(B_1{}^{o}) \right) \subsetneq B_2{}^{o}$. Then $G := \langle G_1, G_2 \rangle = G_1 * G_2$ is a free product, $D := (D_1 \cap B_2) \cup (D_2 \cap B_1)$ is a fundamental domain for G, every non-loxodromic element of G is conjugate to a (non-loxodromic) element of G_1 or G_2, and G is geometrically finite if G_1 and G_2 both are.



In this picture, $\widehat{\mathbb{C}} \setminus D = \left(\widehat{\mathbb{C}} \setminus D_1\right) \cup \left(\widehat{\mathbb{C}} \setminus D_2\right)$.

We apply this several times. The k building blocks are:

- For $t_j = 1$: $D =$ a parallelogram, large enough (i.e. close enough around ∞).
- For $t_j > 1$: $D =$ a hyperbolic $(4t_j)$-gon with angle sum 2π, lying in a disk of radius R ($R >> 1$) and its mirror $(4t_j)$-gon.

This can be chosen so that $\widehat{\mathbb{C}} \setminus D$ fits into any prescribed small neighbourhood of ∞. After conjugation, this means that D can be chosen such that $\widehat{\mathbb{C}} \setminus D$ fits into any prescribed neighbourhood of a given point $z_j \in \widehat{\mathbb{C}}$.

By applying the First Combination Theorem $(k-1)$ times, we get (i), (ii), (iv).

The number of (connected) components of Ω is

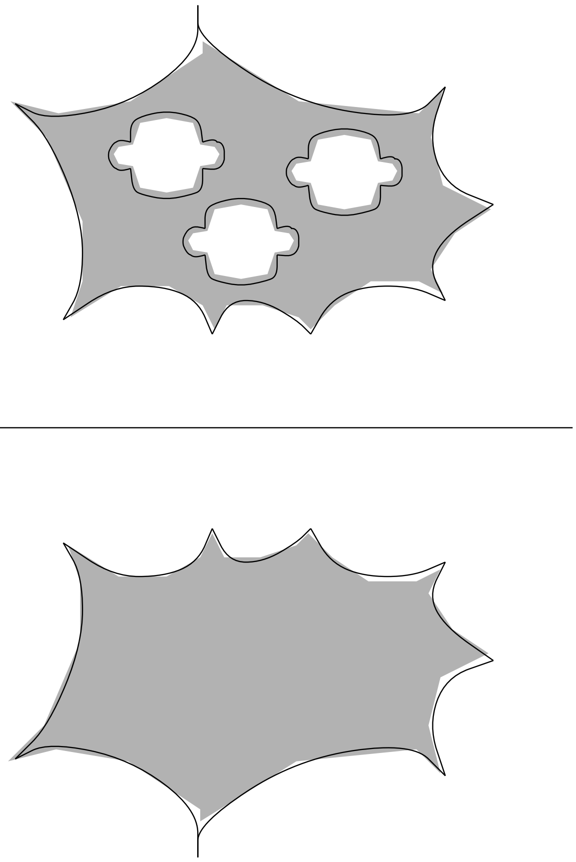
$$\left\{ \begin{array}{ll} 1 & \text{if all } t_j = 1 \\ 2 & \text{if } k = 1 \text{ and } t_1 > 1 \\ \infty & \text{if } k > 1 \text{ and some } t_j > 1 \end{array} \right\}$$

(This is because each factor Δ_{t_j} with $t_j > 1$ adds a circle to the **limit set** $\Lambda(G)$, which is mirrored to other places by $\mathop{\ast}\limits_{i \neq j} \Delta_{t_i}$.)

The number of invariant components of Ω is

$$\left\{\begin{array}{ll} 1 & \text{if } k>1 \text{ or all } t_j=1 \\ 2 & \text{if } k=1 \text{ and } t_1>1 \end{array}\right\}$$

<u>EXAMPLE 68:</u> $G_{t,0}$ with $t=(2,1,1,1)$: The shaded area is a fundamental region for the action of G in Ω. The upper one is the fundamental domain for ${}^{\Omega'}/_{G}$.



Case 2: $s>0$. We use the Second Combination Theorem by Klein and Maskit ([Maskit 88], [Maskit 93]). The theorem proved in [Maskit 88, VII.E], specialized with $J_1=J_2=\{\mathrm{id}\}$, reads as follows:

> COMBINATION THEOREM 2: Let $G_0 \le \mathrm{PGL}_2(\mathbb{C})$ be a discrete group, $G_1 = \langle f \rangle$ cyclic of infinite order; B_1, B_2 two closed disks, such that B_1 and B_2 are disjoint in a fundamental domain D_0 of G_0. Assume that f maps the exterior $\mathrm{ext}\,(B_1)$ of B_1 onto the interior $\mathrm{int}\,(B_2)$ of B_2 and $\mathrm{int}\,(B_1)$ on $\mathrm{ext}\,(B_2)$, and that $A := \mathrm{ext}\,(B_1) \cap \mathrm{ext}\,(B_2)$ satisfies

$A \cap \left(\dot{\bigcup\limits_{g \in G_0}} g(B_1) \dot\cup \dot{\bigcup\limits_{g \in G_0}} g(B_2) \right) \subsetneqq A$. Assume that D_0 is a fundamental domain of ${}^{\circ}\Omega(G_0)/_{G_0}$ with $f(D_0 \cap \partial B_1) = D_0 \cap \partial B_2$. Then $G := \langle G_0, f \rangle = G_0 * \langle f \rangle$ is a free product, $D := D_0 \cap (A \cup \partial B_1)$ is a fundamental domain for ${}^{\circ}\Omega(G)/_{G}$, every non-loxodromic element of G is conjugate to a (non-loxodromic) element of G_0, and G is geometrically finite if G_0 is.

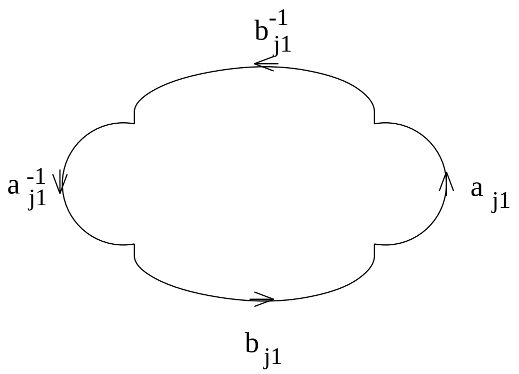
We apply the Second Combination Theorem to $G_0 :\cong G_{t,s-1}$ (induction hypothesis), after choosing two disjoint disks C_s and C'_s in D_0 and an $f \in \mathrm{PGL}_2(\mathbb{C})$ which maps

$$f : \left\{ \begin{array}{c} \mathrm{int}\, C_s \to \mathrm{ext}\, C'_s \\ \partial C_s \to \partial C'_s \\ \mathrm{ext}\, C_s \to \mathrm{int}\, C'_s \end{array} \right\}$$

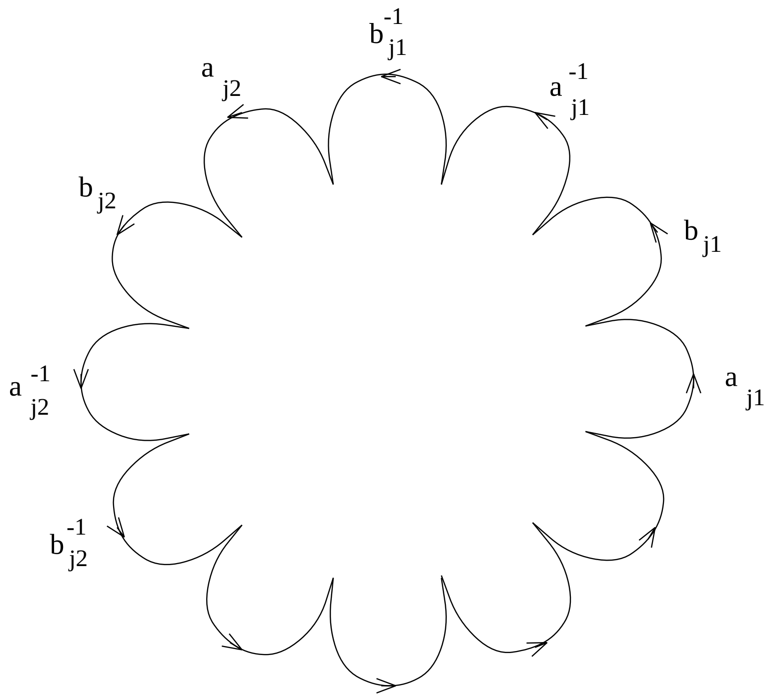
Note that by Note 29, p. 26 $G_{t,s}$ is torsion-free, therefore $G \cong G_{t,s}$ contains no elliptic elements, and ${}^{\circ}\Omega(G) = \Omega(G)$ by [Maskit 88, II.E.8].

This proves (i)-(iv).

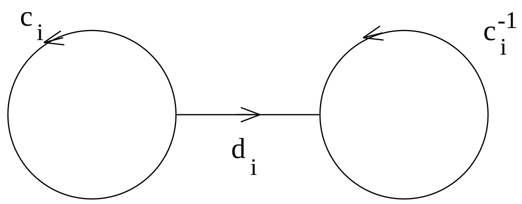
Since ${}^{\circ}\Omega(G) = \Omega(G)$ ($G_{t,s}$ is torsion-free), the covering $\Omega' \to \Omega'/_{G} =: X$ is unramified, and X is a RS, by [Maskit 88, III.A.3]. X is compact because the fundamental domain is relatively compact in Ω'. All that remains to be proved is that $g(X) = r + s$. For this purpose, consider $\pi_1(X)$ as an object of the category **Grp** defined in [Wewers 95, section 2.2.2, p.7]. By (iii) we know that a fundamental domain for $\Omega'/_{G}$ is the common exterior of



$t_j = 1$



$t_j > 1$



$i \in \{1, \dots, s\}$

It follows that

$$\pi_1(X) = \left\langle a_{1,1}, b_{1,1}, \dots, a_{k,t_k}, b_{k,t_k}, c_1, d_1, \dots, c_s, d_s \mid \prod_{j=1}^{k} \prod_{i=1}^{t_j} [a_{j,i}, b_{j,i}] \cdot \prod_{i=1}^{s} [c_i, d_i] = 1 \right\rangle$$

Therefore $g(X) = \dfrac{1}{2} \operatorname{rank}\left(\pi_1(X)^{\text{ab}}\right) = \dfrac{1}{2} \operatorname{rank}\left(\mathbb{Z}^{2r+2s}\right) = r + s.$ **q.e.d.**

The following result of Maskit covers a more general situation than Theorem 67, p. 42. It does, however, not give the geometric details proved in Theorem 67′.

Proposition 69: Let X be a CRS of genus g, and $Y \longrightarrow X$ a regular covering of RS, with Y planar. Then Y can be embedded into $\hat{\mathbb{C}}$ in such a way that (equivalently)

(i) $\forall f \in \mathcal{D}eck(Y/X) : \forall z \in Y : S_f(z) = 0$.

(ii) $\forall f \in \mathcal{D}eck(Y/X) : f$ is the restriction of a Möbius transformation to Y.

(iii) $\mathcal{D}eck(Y/X)$ can be viewed as a Kleinian group.

Proof: [Maskit 88, Theorem X.F16, p. 299].
This proposition is remarkable for two reasons: First, Maskit did not have Theorem 59, p. 39. Second, even in the case $g \geq 2$, $\pi_1(Y) = \langle [a_1, b_1] \rangle_{\Delta_g} \trianglelefteq \pi_1(X)$, $\mathcal{D}eck(Y/X) = {}^{\pi_1(X)}/_{\pi_1(Y)} \cong \Delta_1 * \Delta_{g-1}$, Y is difficult to describe geometrically. Probably $Y \notin O_{AD}$ if $g > 2$.
Even more daring than this proposition is the following

<u>Conjecture 1:</u> Let X be a CRS of genus g, and $Y \longrightarrow X$ a regular covering of RS, with $\mathrm{ir}\,(\mathcal{D}eck(Y/X)) = g$. Then Y can be embedded into $\widehat{\mathbb{C}}$ in such a way that $\mathrm{Aut}_{\mathbf{RS}}(Y) \leq \mathrm{Aut}_{\mathbf{RS}}\left(\widehat{\mathbb{C}}\right) = \mathrm{PGL}_2(\mathbb{C})$.

This conjecture is known to be true if $Y \in O_{AD}$ (see [Ahlfors, Sario 60, IV.2D, p. 200]) or if Y is simply connected (by the uniformization theorem, [Ahlfors, Sario 60, III.11G, p. 181]).

2.5. Coverings of RSN.

<u>Lemma 70:</u> Let $p: Y \to X$ be a covering of RSN.
(a) $y \in Y$ is a node of $Y \Leftrightarrow p(y)$ is a node of X.
(b) If Y_j is a component of Y, then $p(Y_j)$ is a component of X. If X_j is a component of X, then $p^{-1}(X_j)$ is the union of some components of Y.
(c) p induces a covering of the intersection graphs: $\overline{p}: G_Y \to G_X$.

Proof:
(a) From the fact that p is locally homeomorphic.
(b) This follows from the fact that two points x_0, x_1 are in the same component of X if and only if there is a path $\gamma: [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$, $\gamma((0,1)) \subset X \setminus N_X$.
(c) Immediate from (a) and (b).

q.e.d.

<u>Lemma 71:</u> Let X be an RSN. Then

$$\pi_1(X) = \pi_1(G_X) * \mathop{\ast}_{X_j \text{ component of } X} \pi_1(X_j)$$

Proof: If X has only a finite number of components, use Theorem 11, p. 17, claim 2. In the general case, use the fact that $\pi_1(X)$ is the direct limit of $\pi_1(Z)$, where Z is a connected union of a finite number of components X_j of X. **q.e.d.**
Let $p: Y \to X$ be a covering of RSN. Let us examine the injection

$$\begin{array}{c} \pi_1(Y) = \pi_1(G_Y) * \mathop{\ast}_{Y_j} \pi_1(Y_j) \\ \Big\downarrow{\scriptstyle p_*} \\ \pi_1(X) = \pi_1(G_X) * \mathop{\ast}_{X_j} \pi_1(X_j) \end{array}$$

Lemma 72: Let $p: Y \to X$ be a covering of RSN.

(a) We have the commutative diagram

$$\begin{tikzcd}
\pi_1(G_Y) \arrow[r, hook] \arrow[d, "\overline{p}_*"'] & \pi_1(Y) \arrow[r, two heads] \arrow[d, "p_*"'] & \pi_1(G_Y) \arrow[d, "\overline{p}_*"'] \\
\pi_1(G_X) \arrow[r, hook] & \pi_1(X) \arrow[r, two heads] & \pi_1(G_X)
\end{tikzcd}$$

(b) Let Y_j be a component of Y and $X_j := p(Y_j)$ the corresponding component of X. We then have the commutative diagram

$$\begin{tikzcd}
\pi_1(Y) \arrow[r, two heads] \arrow[d, "p_*"'] & \pi_1(Y_j) \arrow[d, "p_*"] \\
\pi_1(X) \arrow[r, two heads] & \pi_1(X_j)
\end{tikzcd}$$

In the Wewers category **Grp** ([Wewers 95, section 2.2.2, p.7]) we also have the commutative diagram

$$\begin{tikzcd}
\pi_1(Y_j) \arrow[r, hook] \arrow[d, "p_*"'] & \pi_1(Y) \arrow[d, "p_*"] \\
\pi_1(X_j) \arrow[r, hook] & \pi_1(X)
\end{tikzcd}$$

Proof: Obvious. **q.e.d.**

This is all we need for Theorem 75, p. 50, but for the sake of section 2.9. we do a more detailed analysis.

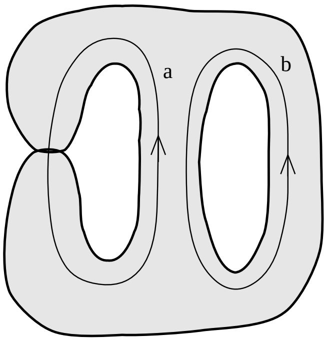
Lemma 73: Let $p: Y \to X$ be a regular covering of RSN, associated to $U \trianglelefteq \pi_1(X)$ (via Definition 50, p. 34). Let X_j be a component of X. For every node $x \in N_{X_j}$, consider the loops γ_x in $\left(X_j \setminus N_{X_j}\right) \cup \{x\}$ which begin and end in x, but through different neighbourhood disks of x. Let X'_j be the RSN X_j with the singularities $\left\{x \in N_{X_j} : [\gamma_x] \notin U \ \ \forall \gamma_x\right\}$ blown-up (replace each of these nodes by two points, one for each neighbourhood disk). The $x \in N_{X_j}$ with $\exists \gamma_x : [\gamma_x] \in U$ are still singularities of X'_j.

(a) For every component $Y_j \subset p^{-1}(X_j)$ of Y:
   (i) We have a regular covering $Y_j \to X'_j$.
   (ii) $\pi_1(Y_j) \trianglelefteq \pi_1\left(X'_j\right) \leq \pi_1(X_j)$.
   (iii) $\pi_1(Y_j) = \pi_1\left(X'_j\right) \cap U$.

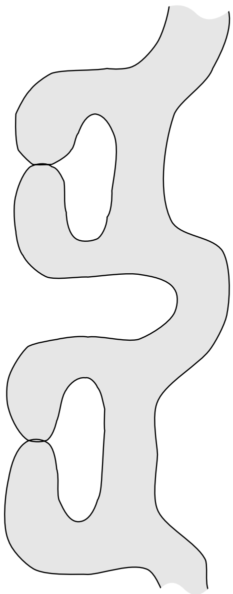
(b) Different choices of Y_j lead to the same $\pi_1(Y_j) \subset \pi_1(X_j)$, but to different (but conjugate) images $\pi_1(Y_j) \subset \pi_1(X)$. (The reason for this is that there are many embeddings $\pi_1(X_j) \hookrightarrow \pi_1(X)$. The embedding suggested by Lemma 71, p. 47, is not canonical.)

Proof: Easy. **q.e.d.**

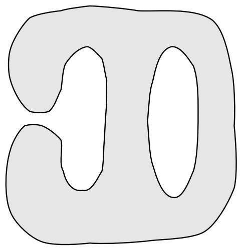
Example 74: Let

$X =$ 

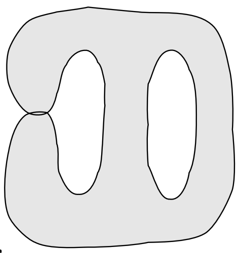
(a) $U = \langle a \rangle_{\pi_1(X)}$ (b) $U = \langle b \rangle_{\pi_1(X)}$



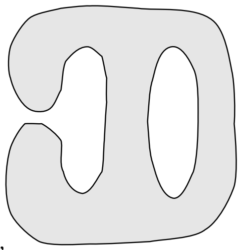
Y_j



Y_j



X_j'



X_j'

2.6. The G-covering of a RSN.
Let $G = G_{t,s}$.

<u>THEOREM 75:</u> For every $X \in \mathcal{M}_{g,G}$, there is a regular covering of RSN

$$\begin{array}{c} Y \\ p\big\downarrow \\ X \end{array}$$

with $\mathcal{D}eck(Y/X) \cong G$. The components of Y are planar RS and form a *tree* of RS, i.e. the intersection graph G_Y is a tree, and

$$\pi_1(Y) = \mathop{\ast}_{Y_j \text{ component of } Y} \pi_1(Y_j)$$

Proof: We are given $\varepsilon : \pi_1(X) \twoheadrightarrow G$. Put $N := \operatorname{kern} \varepsilon \trianglelefteq \pi_1(X)$. Let Y be the regular covering of X associated to the normal subgroup N. By Definition 50, p. 34, and Proposition 52, p. 35, $\mathcal{D}eck(Y/X) \cong {}^{\pi_1(X)}/_{N} \cong G$ and $N = \pi_1(Y)$. By Lemma 71, p. 47, and Lemma 72, p. 48, we have the following surjective homomorphisms:

$$\begin{aligned}
{}^{\pi_1(G_X)}/_{\pi_1(G_Y)} * \mathop{\ast}_{X_j} \pi_1(X_j) &\twoheadrightarrow {}^{\pi_1(G_X)}/_{\pi_1(G_Y)} * \mathop{\ast}_{X_j} \left({}^{\pi_1(X_j)}/_{\langle \pi_1(Y_j) \mid Y_j \subset p^{-1}(X_j) \rangle} \right) \\
&= \left(\pi_1(G_X) * \mathop{\ast}_{X_j} \pi_1(X_j) \right) / \left(\pi_1(G_Y) * \mathop{\ast}_{Y_j} \pi_1(Y_j) \right) \\
&= {}^{\pi_1(X)}/_{\pi_1(Y)} \\
&\cong \mathcal{D}eck(Y/X) \\
&\cong G \\
&\twoheadrightarrow \mathbb{Z}^{*g}
\end{aligned}$$

If there is some $\gamma \in \pi_1(G_Y) \setminus \{1\}$, lemma 22, p. 24 shows that

$$\operatorname{ir} \left({}^{\pi_1(G_X)}/_{\pi_1(G_Y)} * \mathop{\ast}_{X_j} \pi_1(X_j) \right) \leq g(X) - 1 = g - 1$$

which is a contradiction. Therefore $\pi_1(G_Y) = \{1\}$, i.e. G_Y is a tree, and Y is a tree of RSN.
Similarly, consider a component Y_j, and let $X_j := p(Y_j)$. We must have

$$\operatorname{ir} \left({}^{\pi_1(X_j)}/_{\langle \pi_1(Y_j) \rangle} \right) = \operatorname{ir} (\pi_1(X_j))$$

because otherwise the inner rank of

$$\pi_1(G_X) * {}^{\pi_1(X_j)}/_{\langle \pi_1(Y_j), \ldots \rangle} * \cdots$$

will (by Lemma 22, p. 24) be strictly less than

$$\mathrm{ir}\,(\pi_1(G_X)) + \mathrm{ir}\,(\pi_1(X_j)) + \cdots = \mathrm{ir}\,(\pi_1(X)) = g(X)$$

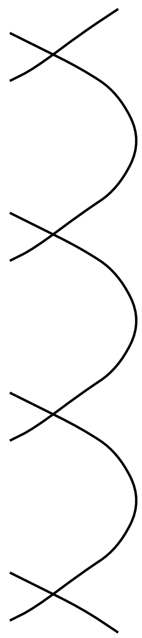
which is impossible.
Now, if Y_j has a node z, there is a loop γ in $Y_j \setminus p^{-1}(N_X) \cup \{z\}$ which begins and ends in z but through different neighbourhood disks of z. The projection $p(\gamma)$ is a loop in $(X_j \setminus N_X) \cup \{p(z)\}$ which begins and ends in different neighbourhood disks of $p(z)$, therefore represents a generator of $\pi_1(X_j)$: $\pi_1(X_j) = \langle p(\gamma)\rangle * \pi_1(X_j')$, where X_j' is a RSN which has one less singularity than X_j. This results in

$$\begin{aligned}\mathrm{ir}\left({}^{\pi_1(X_j)}/_{\langle \pi_1(Y_j)\rangle}\right) \le \mathrm{ir}\left({}^{\pi_1(X_j)}/_{\langle p(\gamma)\rangle_{\pi_1(X_j)}}\right) &= \mathrm{ir}\,(\pi_1(X_j')) = g(X_j') \\ &< g(X_j) = \mathrm{ir}\,(\pi_1(X_j))\end{aligned}$$

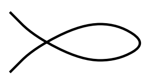
which is impossible. Therefore every component Y_j is a RS without nodes.

EXAMPLE 76:

Y = a tree of projective lines



X with $\pi_1(X) = G_{0,1}$



Let $n_j = \#N_{X_j}$ and X'_j as in section 1.3 and in lemma 73, p. 48. $Y_j \xrightarrow{p} X'_j$ is a UUC of RS, $\pi_1(X_j) = \pi_1(X'_j) * \mathbb{Z}^{*n_j}$, $\pi_1(Y_j) \trianglelefteq \pi_1(X'_j) \leq \pi_1(X_j)$, hence ${\pi_1(X_j)}/{\langle\pi_1(Y_j)\rangle} = {\pi_1(X'_j)}/{\pi_1(Y_j)} * \mathbb{Z}^{*n_j}$. We see that ${\pi_1(X'_j)}/{\pi_1(Y_j)}$ is a free factor of ${\pi_1(X)}/{\pi_1(Y)} \cong \mathcal{D}eck(Y/X) \cong G$, hence itself isomorphic to some $G_{t^{(j)},s^{(j)}}$ by Corollary 41, p. 30. From $\mathrm{ir}\left({\pi_1(X_j)}/{\langle\pi_1(Y_j)\rangle}\right) = \mathrm{ir}\left(\pi_1(X_j)\right)$ we get $\mathrm{ir}\left({\pi_1(X'_j)}/{\pi_1(Y_j)}\right) = \mathrm{ir}\left(\pi_1(X'_j)\right)$. By the Planarity Theorem 59, p. 39, it follows that Y_j is planar. The rest follows from Lemma 71, p. 47. **q.e.d.**

2.7. A lemma about graphs.

In this section, a **graph** is understood to be an undirected graph, which is viewed as a directed graph by replacing each undirected edge by two (distinct) directed edges in opposite directions. (See [Cohen 89, chapter 5.1, p. 113] for a formal definition.)
Let A, C be graphs and $p : A \to C$ a map of graphs.
For every vertex v, let E_v be the set of edges starting at v. For every vertex v of A, we have a map $p_v : E_v \to E_{p(v)}$. p is called **locally surjective** if $\forall v \in A : p_v : E_v \to E_{p(v)}$ surjective. p is called **locally injective** if $\forall v \in A : p_v : E_v \to E_{p(v)}$ injective. (See [Cohen 89, p. 184].)

<u>Definition 77:</u> For every vertex w of C, let a non-empty set N_w be given. Put $N := (N_w)_{w\in C}$. We call p **locally N-fold** if for every $v \in A$, $w := p(v) \in C$ and every $e \in E_w$ there is a bijection between ${p_v}^{-1}(e) = p^{-1}(e) \cap E_v$ and N_w. This implies that p is locally surjective and that for every $v \in A$, there is a bijection between E_v and $E_w \times N_w$ ($w := p(v)$).

The following lemma generalizes [Cohen 89, chapter 8.1, Lemma 2 and Lemma 4, p. 184].

<u>Lemma 78:</u> Let A, B, C be graphs, $p : A \to C$ and $q : B \to C$ locally N-fold maps of graphs. Assume that A and B are trees. Then $A \cong B$. More precisely, given $v_A \in A$ and $v_B \in B$ with $p(v_A) = q(v_B)$, there is an isomorphism $\varphi : A \to B$ with $\varphi : v_A \mapsto v_B$, $q \circ \varphi = p$.

Proof: Consider

$$\left\{\begin{aligned}(T,\varphi) :&T \text{ subtree of } A, v_A \in T,\\ &\varphi : T \to B \text{ a map of graphs}, \varphi(v_A) = v_B,\\ &\{\text{edges of } T\} \text{ is the disjoint union of some } {p_v}^{-1}(e)\ (v \in A, e \in E_{p(v)}),\\ &\{\text{edges of } \varphi(T)\} \text{ is the disjoint union of some } {q_v}^{-1}(e)\ (v \in B, e \in E_{q(v)})\end{aligned}\right\}$$

Such pairs exist: for example $(T = \{v_A\}, \varphi : v_A \mapsto v_B)$. This set has a partial ordering, given by $(T_1,\varphi_1) \leq (T_2,\varphi_2) \;:\Leftrightarrow\; \left(T_1 \subset T_2 \;\wedge\; \varphi_1 = \varphi_2|_{T_1}\right)$. By Zorn's lemma, there is a maximal (T,φ). We claim that $T = A$.
If there is some vertex $\in A \setminus T$, there is an edge f starting at a point $v \in T$ and ending at a point $\in A \setminus T$. Put $e := p(f)$ and adjoin all the edges $\in {p_v}^{-1}(e)$ to T (they are not yet in T since f is not) and all the edges $\in {q_{\varphi(v)}}^{-1}(e)$ to $\varphi(T)$ (they are not yet in $\varphi(T)$ since $q \circ \varphi = p$). Since $\#{p_v}^{-1}(e) = \#N_{p(v)} = \#{q_{\varphi(v)}}^{-1}(e)$, it is possible to extend φ this

way. This contradicts the maximality of (T, φ). Therefore T has the same vertex set as A. Since A is a tree and T is connected, $T = A$.
For every edge e of C, beginning at some point $p(v)$, we have a bijection

$$\varphi_{v,e} : {p_v}^{-1}(e) \to {q_{\varphi(v)}}^{-1}(e)$$

Taking the union over all $e \in E_{p(v)}$, we get a bijection

$$\varphi_v : E_v = \dot{\bigcup_{e \in E_{p(v)}}} {p_v}^{-1}(e) \to \dot{\bigcup_{e \in E_{p(v)}}} {q_{\varphi(v)}}^{-1}(e) = E_{\varphi(v)}$$

Therefore φ is locally bijective, and since locally bijective maps between trees are bijective ([Cohen 89, chapter 8.1, Lemma 5(iii), p. 185]), φ is bijective. **q.e.d.**

2.8. Graphs of groups.

DEFINITION 79: A **restricted graph of groups (RGG)** $\mathcal{G}$ is a connected graph $\mathcal{G}^o = (V, E)$ with a group G_v attached to every vertex $v \in V$. (This is the same as a graph of groups with trivial edge groups, in the sense of [Cohen 89, chapter 8.3, p. 198].)

DEFINITION 80: The **fundamental group** $\pi(\mathcal{G})$ of a RGG $\mathcal{G}$ is defined as

$$\pi(\mathcal{G}) := \pi_1(\mathcal{G}^o) * \mathop{\Huge *}_{v \in V} G_v$$

(See [Cohen 89, chapter 8.3, p. 199].)

DEFINITION 81: The **universal cover** of a RGG $\mathcal{G}$ is the graph (*not* RGG!) whose vertex set is

$$\{(g, v) : g \in \pi(\mathcal{G}), v \in V\}/_\sim$$

where $(g_1, v) \sim (g_2, v)$ if ${g_1}^{-1} g_2 \in G_v$, and which has an edge $[(g, v_1)]$—$[(g, v_2)]$ for every edge v_1—v_2 in $\mathcal{G}$ and every $g \in \pi(\mathcal{G})$. (See [Cohen 89, chapter 8.4, p. 205] or [Serre 77].)

2.9. The tree.

We keep the notations of the proof of Theorem 75, p. 50.

<u>DEFINITION 82:</u> For $X \in \mathcal{M}_{g,G}$ let $\mathcal{G}_X$ be the following restricted graph of groups: The vertices are the components X_j of X. For every node connecting a component X_j to a component X_i, add an edge X_j—X_i to $\mathcal{G}_X$. (Here, the nodes of X_j count as edges, in contrast to the intersection graph G_X.) For every vertex X_j, choose a component Y_j of Y lying above X_j, and attach $\mathcal{D}eck(Y_j/X'_j)$ to X_j.

This is well-defined because of Lemma 73, p. 48.

NOTE 83: The fundamental group $\pi(\mathcal{G}_X)$ is

$$\begin{aligned}
\pi(\mathcal{G}_X) &= \pi_1(\mathcal{G}_X{}^{o}) * \left(\mathop{\ast}_{X_j} \mathcal{D}eck\big(Y_j/X_j'\big) \right) \\
&= \pi_1(G_X) * \left(\mathop{\ast}_{X_j} \mathbb{Z}^{*(\#N_{X_j})} \right) * \left(\mathop{\ast}_{X_j} \mathcal{D}eck\big(Y_j/X_j'\big) \right) \\
&= \pi_1(G_X) * \mathop{\ast}_{X_j} \left({}^{\pi_1(X_j)}\big/_{\langle \pi_1(Y_j) \rangle} \right) \\
&= {}^{\pi_1(X)}\big/_{\pi_1(Y)} \\
&\cong \mathcal{D}eck(Y/X) \\
&\cong G
\end{aligned}$$

THEOREM 84: G_Y is the universal cover of $\mathcal{G}_X$.

Proof: The universal cover $\widetilde{\mathcal{G}_X}$ of $\mathcal{G}_X$ is a tree, by [Cohen 89, chapter 8.4, Theorem 24, p. 208]. G_Y is a tree by Theorem 75, p. 50. It is not difficult to see that

$$p : \left\{ \begin{array}{r} \widetilde{\mathcal{G}_X} \to \mathcal{G}_X{}^{o} \\ [(g,v)] \mapsto v \\ [(g,v_1)] — [(g,v_2)] \mapsto v_1 — v_2 \end{array} \right\}$$

and

$$q : \left\{ \begin{array}{ll} G_Y \to & \mathcal{G}_X{}^{o} \\ Y_j \mapsto & X_j \text{ below } Y_j \end{array} \right\}$$

both are locally $\mathcal{G}_X$-fold: every node of X, connecting X_j to some X_i, has $\#\mathcal{D}eck\big(Y_j/X_j'\big)$ preimages in a fixed component Y_j lying above X_j. Now apply Lemma 78, p. 52. **q.e.d.**
Note: If X is a RS, i.e. $\mathcal{G}_X{}^{o}$ is the trivial graph (consisting of a single vertex and no edges), it is easily seen that $\widetilde{\mathcal{G}_X}$ is the trivial graph as well, regardless of the group $\mathcal{D}eck(Y/X)$. Of course, we already knew that G_Y is the trivial graph if and only if X has no nodes.

3. Teichmüller spaces

In this chapter, we give a summary of well-known facts about Teichmüller spaces of Riemann surfaces, then prove that the deformation space of $G_{t,s}$ as a Kleinian group is a finite-dimensional complex analytic manifold, then exhibit global holomorphic coordinates for this deformation space.

3.1. The classical Teichmüller theory for Riemann surfaces.

In this section, we merely give an outline of the classical Teichmüller theory. Given a RS X, the problem is to classify the **conformal structures** on X. There are three approaches:

1. Fricke coordinates. See [Abikoff 76, § I.1.2].
2. Fenchel-Nielsen coordinates (pairs of pants). See [Abikoff 76, § II.3].
3. Use the theory of **quasiconformal (qc) mappings**. (For compact X, they can be avoided to a certain extent, being replaced by diffeomorphisms.) Via the Beltrami differential equation, one establishes that the conformal structures on X correspond to **complex dilatations** μ of qc mappings. For compact X, the Riemann-Roch theorem and its consequences then give that conformal structures on X correspond to certain **quadratic differentials** on X.

We will outline the third approach. This material can be found e.g. in [Abikoff 76], [Nag 88], [Earle 77], [Bers 72], [Imayoshi, Taniguchi 92].

The starting point of the theory is the **uniformization theorem**: For every RS X, the universal covering $\tilde{X}$ is conformally equivalent to either $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$. And the question: If X depends holomorphically on some parameters, will $p: \tilde{X} \to X$ depend on these parameters holomorphically as well? Recall that, for $g = 1$,

$$p : \left\{ \begin{array}{ll} \mathbb{C} \to \mathbb{C}/_{\Lambda_\tau} = E_\tau \subset \mathbb{P}^2(\mathbb{C}) \\ z \mapsto \quad (\wp_\tau(z), \wp'_\tau(z)) \end{array} \right\}$$

is holomorphic in τ. More precisely, the question ([Bers 60, Theorem 4, p. 97]): Are there meromorphic functions $p_1(z,\tau), \dots, p_N(z,\tau)$ on $\mathbb{D} \times \mathcal{M}_g$, such that for every $\tau \in \mathcal{M}_g$: $p_1(\cdot,\tau), \dots, p_N(\cdot,\tau)$ generate the field of meromorphic functions of X_τ?

One defines

$$\mathcal{M}_g = \{\text{compact RS of genus } g\}/_{\text{conformal equivalence}}$$

Fix a compact RS X of genus g and consider all conformal atlases on X (not just one). One has

$$\mathcal{M}_g = \{(X, \mathfrak{A}) : \mathfrak{A} \text{ conformal atlas on } X\}/_{\text{conformal equivalence}}$$

(because any two RS of genus g are topologically equivalent, and the atlas can be transported from one RS to the other via a homeomorphism).

Fix $m := \max(3 - 2g, 0)$ distinct points $x_1, \dots, x_m$ on X. Since any m distinct points on X can be mapped to $(x_1, \dots, x_m)$ by a conformal automorphism of X, we will define X to mean $(X; x_1, \dots, x_m)$ from now on, and still have:

$$\mathcal{M}_g = \{(X, \mathfrak{A}) : \mathfrak{A} \text{ conformal atlas on } X\}/_{\text{conformal equivalence}}$$

An automorphism of X is understood to fix $x_1, \dots, x_m$ from now on.

The following lemma ([Abikoff 76, Proposition 2, p. 5]):

LEMMA 85: If $f : (X, \mathfrak{A}) \to (X, \mathfrak{A})$ is a biholomorphic automorphism, and $f \approx \mathrm{id}_X$ (i.e. f is **homotopic** to the identity), then $f = \mathrm{id}_X$.

implies that

$$\mathcal{M}_g = \{(X, \mathfrak{A}) : \mathfrak{A} \text{ conformal atlas on } X\} / \left\{ \begin{array}{l} f : X \to X \text{ homeomorphism,} \\ f \approx g, \\ g \text{ conformal homeomorphism} \end{array} \right\}$$

or

$$\mathcal{M}_g = \{(X, \mathfrak{A}) : \mathfrak{A} \text{ conformal atlas on } X\} / \left\{ \begin{array}{l} f = g \circ h, \\ g \text{ conformal homeomorphism,} \\ h \text{ homeomorphism,} \\ h \approx \mathrm{id} \end{array} \right\}$$

where $\approx$ means homotopy or isotopy (equivalently, by the theorem of Baer-Mangler-Epstein).
Now fix an atlas $\mathfrak{A}_0$ on X and put $S = (X, \mathfrak{A}_0)$. Since every $f \in \mathrm{Aut}_{\mathcal{C}}(X)$ is homotopic to some $\in \mathrm{Aut}_{\mathcal{C}^\infty}(X)$, we have

$$\mathcal{M}_g = \left\{ \begin{array}{rl} (X, \mathfrak{A}) : & \mathfrak{A} \text{ conformal atlas on } X \\ & \mathcal{C}^\infty \text{ relative to } \mathfrak{A}_0 \end{array} \right\} / \left\{ \begin{array}{l} f : (X, \mathfrak{A}_1) \to (X, \mathfrak{A}_2) \\ \mathcal{C}^\infty \text{ automorphism,} \\ f \approx g, \\ g \text{ conformal homeomorphism} \end{array} \right\}$$

(This approach makes sense only for compact X. In general, use the class of qc automorphisms instead of $\mathcal{C}^\infty$ automorphisms.)
Since for every atlas $\mathfrak{A}$ there is the conjugate atlas $\overline{\mathfrak{A}} = \{\overline{z} : z \text{ chart in } \mathfrak{A}\}$, we have

$$\mathcal{M}_g = \left\{ \begin{array}{rl} (X, \mathfrak{A}) : & \mathfrak{A} \text{ conformal atlas on } X \\ & \mathcal{C}^\infty \text{ relative to } \mathfrak{A}_0 \\ & \text{same orientation as } \mathfrak{A}_0 \end{array} \right\} / \left\{ \begin{array}{l} f : (X, \mathfrak{A}_1) \to (X, \mathfrak{A}_2) \\ \text{orientation preserving} \\ \mathcal{C}^\infty \text{ automorphism,} \\ f \approx g, \\ g \text{ conformal homeomorphism} \end{array} \right\} \tag{1}$$

One can prove (cf. [Abikoff 76, p. 67] ??) that every orientation preserving $\mathcal{C}^\infty$ automorphism of X is homotopic to a conformal homeomorphism. Therefore

$$\mathcal{M}_g = \left\{ \begin{array}{rl} (X, \mathfrak{A}) : & \mathfrak{A} \text{ conformal atlas on } X \\ & \mathcal{C}^\infty \text{ relative to } \mathfrak{A}_0 \\ & \text{same orientation as } \mathfrak{A}_0 \end{array} \right\} / \left\{ \begin{array}{l} f : (X, \mathfrak{A}_1) \to (X, \mathfrak{A}_2) \\ \text{orientation preserving} \\ \mathcal{C}^\infty \text{ automorphism} \end{array} \right\}$$

Note that in $f : (X, \mathfrak{A}_1) \to (X, \mathfrak{A}_2)$ the atlases $\mathfrak{A}_1, \mathfrak{A}_2$ don't matter any more, except for their $\mathcal{C}^\infty$ structure and their orientation, which they share with $\mathfrak{A}_0$. We therefore define

$$\mathrm{Diff}^+(X) := \{f : (X, \mathfrak{A}_1) \to (X, \mathfrak{A}_2) \text{ orientation preserving } \mathcal{C}^\infty \text{ automorphism}\}$$

$$\mathrm{Diff}_0(X) := \{f : X \to X\ \mathcal{C}^\infty \text{ automorphism}, f \approx \mathrm{id}_X\} \trianglelefteq \mathrm{Diff}^+(X)$$

The **mapping class group** is

$$\mathrm{Mod}_g := {}^{\mathrm{Diff}^+(X)}\!/_{\mathrm{Diff}_0(X)}$$

The **Teichmüller space** of X is

$$T_g := \left\{ \begin{array}{rl} (X, \mathfrak{A}) : & \mathfrak{A} \text{ conformal atlas on } X \\ & \mathcal{C}^\infty \text{ relative to } \mathfrak{A}_0 \\ & \text{same orientation as } \mathfrak{A}_0 \end{array} \right\} /_{\mathrm{Diff}_0(X)} \tag{2}$$

We now have

$$\mathcal{M}_g = {}^{T_g}\!/_{\mathrm{Mod}_g}$$

The theorem of Dehn-Nielsen describes Mod_g explicitly. First some notations:
If $f : X \to Y$ is some continuous map between path-connected spaces, we denote $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ by $\pi_1(f)$.
If G is a group, $\mathrm{Out}\,(G) := {}^{\mathrm{Aut}\,(G)}\!/_{\mathrm{Inn}\,(G)}$ is the automorphism group modulo inner automorphisms.
If G is a group with $G^{\mathrm{ab}} \cong \mathbb{Z}^n$, define
$\mathrm{Aut}^+(G) := \left\{\varphi \in \mathrm{Aut}\,(G) : \varphi^{\mathrm{ab}} \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^n) \text{ has determinant } +1, \text{ not } -1\right\}$ and
$\mathrm{Out}^+(G) := {}^{\mathrm{Aut}^+(G)}\!/_{\mathrm{Inn}\,(G)}$.
Recall that $\pi_1(X) \cong \Delta_g$, $\pi_1(X)^{\mathrm{ab}} \cong \mathbb{Z}^{2g}$.

Theorem of Dehn-Nielsen: There is a group isomorphism

$$\left\{ \begin{array}{ccc} \mathrm{Mod}_g & \to & \mathrm{Out}^+(\pi_1(X)) \\ f \cdot \mathrm{Diff}_0(X) & \mapsto & \pi_1(f) \cdot \mathrm{Inn}\,(\pi_1(X)) \end{array} \right\}$$

It follows that

$$T_g = \left\{ \begin{array}{rl} (S, \tau) : & S = {}^{\tilde{X}}\!/_{G}, \\ & \tau : \Delta_g \to G \text{ a group isomorphism to a subgroup } G \leq \mathrm{Aut}_{\mathbf{RS}}\left(\tilde{X}\right) \end{array} \right\}$$

or

$$T_g = \left\{ \begin{array}{rl} \tau : \Delta_g \to G \leq \mathrm{Aut}_{\mathbf{RS}}\left(\tilde{X}\right) : & \tau \text{ injective group homomorphism}, \\ & G = \mathrm{im}\,\tau \text{ operates discontinuously on } \tilde{X}, \\ & {}^{\tilde{X}}\!/_{G} \text{ a compact RS} \end{array} \right\}$$

Now let us go back to (1) and investigate equality between two atlases.
Definition: Let $(X_1, \mathfrak{A}_1)$, $(X_2, \mathfrak{A}_2)$ be two RS, $f : (X_1, \mathfrak{A}_1) \to (X_2, \mathfrak{A}_2)$ an orientation preserving $\mathcal{C}^\infty$ automorphism. Define the **complex dilatation** $\mu_f \in \mathcal{E}^{-1,1}(X_1)$ as follows: Let $p \in X_1$ a point, $z \in \mathfrak{A}_1$ a chart at p, $w \in \mathfrak{A}_2$ a chart at $f(p)$. Put

$\mu_f(p) := \dfrac{\partial_{\overline{z}}\big(w \circ f \circ z^{-1}\big)}{\partial_z\big(w \circ f \circ z^{-1}\big)}(p) \cdot \dfrac{d\overline{z}}{dz}$ (chart independent!). $|\mu_f(p)| < 1$ since f is orientation preserving.
Define $\mu_{\mathfrak{A}_1}^{\mathfrak{A}_2} := \mu_{\mathrm{id}_X}$, belonging to $\mathrm{id}_X : (X, \mathfrak{A}_1) \to (X, \mathfrak{A}_2)$.
It is clear that: $\mathfrak{A}_1 = \mathfrak{A}_2 \Leftrightarrow \mathrm{id}_X : (X, \mathfrak{A}_1) \to (X, \mathfrak{A}_2)$ is conformal $\Leftrightarrow \mu_{\mathfrak{A}_1}^{\mathfrak{A}_2} = 0$.
For $S = (X, \mathfrak{A})$ define $\mu_S := \mu_{\mathfrak{A}_0}^{\mathfrak{A}} \in B_1\big(\mathcal{E}^{-1,1}(X)\big) := \big\{\mu \in \mathcal{E}^{-1,1}(X) : \|\mu\|_\infty < 1\big\}$. The group $\mathrm{Diff}(X)$ of diffeomorphisms of X operates on $\mathcal{E}^{-1,1}(X)$ like this: For $f \in \mathrm{Diff}(X)$ and $\mu \in \mathcal{E}^{-1,1}(X)$ define $\mu \circ f := f^*\mu = \Big(p \mapsto \frac{f_{\overline{z}}(p) + \mu(f(p)) \cdot \overline{f_z}(p)}{f_z(p) + \mu(f(p)) \cdot \overline{f_{\overline{z}}}(p)} \cdot \frac{d\overline{z}}{dz}\Big)$.
Note: In $S_0 \xrightarrow{f} S_1 \xrightarrow{g} S_2$, we have $\mu_g \circ f = \mu_{g \circ f}$.
Note: In $S_0 \xrightarrow{f} S_1 \xrightarrow{g} S_2$, g is conformal if and only if $\mu_{g \circ f} = \mu_f$.
Proof: It is sufficient to prove this locally. The result is obvious if S_0, S_1, S_2 are domains in $\mathbb{C}$. **q.e.d.**
Note: In $S_0 \xrightarrow{f} S_1 \xrightarrow{g} S_2$, we have $\|\mu_{g \circ f}\|_\infty \leq \dfrac{\|\mu_f\|_\infty + \|\mu_g\|_\infty}{1 + \|\mu_f\|_\infty \cdot \|\mu_g\|_\infty} < 1$, ergo
$\operatorname{artanh} \|\mu_{g \circ f}\|_\infty \leq \operatorname{artanh} \|\mu_f\|_\infty + \operatorname{artanh} \|\mu_g\|_\infty$.
The following theorem is central:

Theorem about the Beltrami differential equation: Given $\mu \in B_1\big(\mathcal{E}^{-1,1}(X)\big)$, there is exactly one conformal atlas $\mathfrak{A}$ on X with $\mu_{\mathfrak{A}_0}^{\mathfrak{A}} = \mu$.
Therefore $S = (X, \mathfrak{A})$ can be identified with μ_S; conformally equivalent surfaces S correspond to equal μ_S. We now have

$$\mathcal{M}_g = B_1\big(\mathcal{E}^{-1,1}(X)\big) \Big/ \left\{\begin{array}{l} f : X_{\mu_1} \to X_{\mu_2}, \\ f \text{ orientation preserving } \mathcal{C}^\infty \text{ automorphism,} \\ f \approx g, \\ g : X_{\mu_1} \to X_{\mu_2} \text{ conformal (i.e. } \mu_1 = \mu_2 \circ g) \end{array}\right\}$$

Writing X as $\tilde{X}/G$, we get from (2):

$$T_g = B_1\left(\mathcal{E}^{-1,1}\left(\tilde{X}\right)^G\right) \Big/ \mathrm{Diff}^+\left(\tilde{X}\right)^G$$

Proof: (Cf. [Earle 77, p. 148-150].) Let $p : \tilde{X} \to \tilde{X}/G$ the canonical map. We then have a bijection

$$\left\{\begin{array}{ccc} \mathcal{E}^{-1,1}(X) & \to & \mathcal{E}^{-1,1}\left(\tilde{X}\right)^G \\ \mu & \mapsto & \mu \circ p \end{array}\right\}$$

and a homomorphism

$$\Phi : \left\{\begin{array}{ccc} \mathrm{Diff}^+\left(\tilde{X}\right)^G & \to & \mathrm{Diff}^+(X) \\ f & \mapsto & pfp^{-1} \end{array}\right\}$$

One proves that
(i) Φ is injective,

(ii) $\operatorname{im}\Phi = \operatorname{Diff}_0(X)$,

using the following

Lemma: $f \in \operatorname{Diff}^+\left(\tilde{X}\right)^G$ can be extended to an $\overline{f} \in \operatorname{Aut}_{\mathrm{qc}}\left(\overline{\tilde{X}}\right)$, and every point of $\partial\tilde{X}$ is a fixed point of $\overline{f}$. **q.e.d.**

3.2. The classical Teichmüller theory for Kleinian groups.

In this section, we give a summary of the Teichmüller theory for Kleinian groups, i.e. the theory of deformation spaces of Kleinian groups. We closely follow the exposition in [Kra 74].

3.2.1. Preliminaries.

Let G be a finitely generated, non-elementary Kleinian group, Ω its region of discontinuity and Λ its limit set. (See [Farkas, Kra 80, chapter IV.5.]. For general background about Kleinian groups, see [Maskit 88].) There is the deep

Theorem 86 of Ahlfors: If G is a finitely generated Kleinian group, then $\Omega(G)/G$ consists of finitely many RS, each of **finite type**, i.e. each a CRS (of finite genus) minus finitely many points called **punctures** with finitely many branch points.

See [Greenberg 77, Theorem 2.1.1., p. 223].

There are quantitative versions of this theorem: Assuming G torsion-free and generated by N generators, the components of the URS $\Omega(G)/G$ are CRS of genus g_i with b_i punctures, respectively, (and no branch points, of course), and $\sum_i (2g_i + b_i - 2) \leq 2(N-1)$. See [Marden 74, p. 397].

Moreover there is a finite-sided fundamental region for the action of G in $\Omega(G)$, see [Greenberg 77, Theorem 2.5.7., p. 242].

3.2.2. The deformation space.

Let $W \subset \Omega$, $W \neq \emptyset$ be a G-invariant union of components of Ω, i.e. W/G the union of some of the RS that constitute $\Omega(G)/G$.

Definition 87: The set of **Beltrami coefficients** is

$$\begin{aligned} M(G,W) &:= B_1\left(L_\infty^{-1,1}(W \to \mathbb{C})^G\right) \\ &= \left\{\mu \in L_\infty^{-1,1}(W \to \mathbb{C}) : \mu\ G\text{-invariant}, \|\mu\|_\infty < 1\right\} \end{aligned}$$

μ G-invariant means $\mu \circ g = \mu\ \forall g \in G$ or, equivalently, $\mu(g(z)) \cdot \dfrac{\overline{g'(z)}}{g'(z)} = \mu(z)\ \forall z \in \widehat{\mathbb{C}}$ $\forall g \in G$.

μ is defined in W, but we will consider it defined on $\widehat{\mathbb{C}}$, extended by $\mu|_{\widehat{\mathbb{C}} \setminus W} := 0$.

Let $a_1, a_2, a_3 \in \Lambda$ three distinct points.

Theorem 88 about the Beltrami equation: There is a bijection

$$\left\{\begin{array}{c} B_1\big(L_\infty^{-1,1}(W \to \mathbb{C})\big) \to \left\{f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f \text{ qc.}, f(a_i) = a_i \quad \forall i \in \{1,2,3\}\right\} \\ \mu \mapsto w^\mu \\ \mu_f := \frac{f_{\bar z}}{f_z} \leftarrow\!\shortmid f \end{array}\right\}$$

Note 89: For an $f \mathrel{\widehat{=}} \mu_f \in M(\{1\}, W)$ we have:

$$\mu_f \ G\text{-invariant} \ \Leftrightarrow \ fgf^{-1} \text{ is conformal } \forall g \in G$$

Corollary 90: For $\mu \in M(G, W)$,

$$G^\mu := \left\{w^\mu \circ g \circ w^{\mu-1} : g \in G\right\}$$

is a Kleinian group, $\Omega(G^\mu) = w^\mu(\Omega)$.

Corollary 91: There is a bijection

$$\left\{\begin{array}{c} M(G,W) \to \left\{\begin{array}{c} f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f \text{ qc.}, f(a_i) = a_i \quad \forall i \in \{1,2,3\}, \\ f|_{\widehat{\mathbb{C}} \setminus W} \text{ conformal}, \\ f \circ g \circ f^{-1} \in \mathrm{PGL}_2(\mathbb{C}) \quad \forall g \in G \end{array}\right\} \\ \mu \mapsto w^\mu \end{array}\right\}$$

Definition 92: The set of **trivial Beltrami coefficients** is

$$M_{\text{triv}}(G,W) := \left\{\begin{array}{c} f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f \text{ qc.}, f(a_i) = a_i \quad \forall i \in \{1,2,3\}, \\ f|_{\widehat{\mathbb{C}} \setminus W} \text{ conformal}, \\ f \circ g \circ f^{-1} = g \quad \forall g \in G \end{array}\right\} \subset M(G,W)$$

Definition 93: $M_{\text{triv}}(G,W)$ operates on $M(G,W)$ via

$$\mu \in M(G,W), f \in M_{\text{triv}}(G,W) \ \Rightarrow \ \mu \circ f \in M(G,W)$$

Definition 94: The **deformation space** is defined as

$$T(G,W) := {}^{M(G,W)}\!/\!{}_{M_{\text{triv}}(G,W)}$$

(Note that [Kra 74, p. 50] denotes it by $\widehat{T}(G,W)$.)

Proposition 95:

$$M_{\mathrm{triv}}(G,W)=\left\{\begin{array}{l} f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}\mid f \text{ qc.},\\ \qquad\qquad f|_{\widehat{\mathbb{C}}\setminus W} \text{ conformal},\\ \qquad\qquad f|_{\Lambda}=\mathrm{id}_\Lambda,\\ \qquad\qquad f\circ g\circ f^{-1}=g \quad \forall g\in G \end{array}\right\}$$

Proof: $f\in M_{\mathrm{triv}}(G,W)$ commutes with every $g\in G$, therefore fixes the fixed points of the loxodromic and parabolic elements of G. Since the limit set Λ is the closure of the set of these fixed points (see [Bers 72, § 1.3., p. 261-262]), $f|_{\Lambda}=\mathrm{id}_\Lambda$. **q.e.d.**

Proposition 96: $M_{\mathrm{triv}}(G,W)=\left\{f\in M(G,W): f|_{\Lambda}=\mathrm{id}_\Lambda\right\}$.

Proposition 97: $f\in M_{\mathrm{triv}}(G,W) \Rightarrow f$ maps every component of Ω onto itself.

Example 98: Let G be a Fuchsian group of the first kind, and $W=\mathbb{H}$. Then

$$T(G,W)=B_1\left(L_\infty^{-1,1}(\mathbb{H})^G\right)/\mathrm{Aut}_{\mathrm{qc}}(\mathbb{H})^G=B_1\left(\mathcal{E}^{-1,1}\left(\tilde{X}\right)^G\right)/\mathrm{Diff}^+\left(\tilde{X}\right)^G=T_g$$

is the classical Teichmüller space, as defined in section 3.1.

3.2.3. The moduli space of deformations.

Consider pairs (F,θ) with

(i) $F:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ qc, $\mu_F\in M(G,W)$ (i.e. $F=F_1\circ F_2$ with $F_1\in\mathrm{PGL}_2(\mathbb{C})$, $F_2\in M(G,W)$).

(ii) The group homomorphism

$$\theta=\theta_F:\left\{\begin{array}{l} G\to\ \mathrm{PGL}_2(\mathbb{C})\\ g\mapsto F\circ g\circ F^{-1}\end{array}\right\}$$

is an isomorphism $\theta: G\to G$.

Note 99: Such an (F,θ) induces a biholomorphic bijection

$$\phi:\left\{\begin{array}{lcl} M(G,W) & \to & M(G,W)\\ \mu & \mapsto & \mu\circ F^{-1}\end{array}\right\}$$

Corollary 100: Such an (F,θ) induces a bijection $\phi: T(G,W)\to T(G,W)$.

We have a subgroup series $\{(F,\theta):\text{(i), (ii)}\}\geq\left\{(F,\theta):F\in\mathrm{PGL}_2(\mathbb{C}),FGF^{-1}=G\right\}\geq\{(F,\theta):F\in G\}\cong G$. For (F,θ) with $F\in G$, the bijection ϕ induced on $M(G,W)$ and $T(G,W)$ is the identity. We therefore define

Definition 101:

$$\mathrm{Mod}(G,W):=\left\{(F,\theta):F\in\mathrm{PGL}_2(\mathbb{C}),FGF^{-1}=G\right\}/\{(F,\theta):F\in G\}$$

and immediately have

NOTE 102: $\mathrm{Mod}(G,W)$ operates on $M(G,W)$ and $T(G,W)$.

DEFINITION 103: The **moduli space of deformations** is

$$\mathcal{M}(G,W) := {}^{T(G,W)}\!/_{\mathrm{Mod}(G,W)}$$

EXAMPLE 98 (CONTINUED): F corresponds to an automorphism of $X := {}^{W}\!/_{G}$ which induces θ. Therefore $\mathrm{Mod}(G,W) = \mathrm{Out}^+(G) = \mathrm{Out}^+(\pi_1(X))$. We have a bijection

$$\left\{\begin{array}{ccc} \mathcal{M}(G,W) & \to & \mathcal{M}(X) \\ [\mu] & \mapsto & {}^{\Omega^+(G^\mu)}\!/_{G^\mu} := {}^{w^\mu(\mathbb{H})}\!/_{G^\mu} \end{array}\right\}$$

onto the moduli space $\mathcal{M}(X) = \mathcal{M}_g$, as defined in section 3.1.

3.2.4. Simultaneous uniformization.

By the Theorem 86, p. 59 of Ahlfors, ${}^{\Omega(G)}\!/_{G} = X_1 \dot\cup \cdots \dot\cup X_l$, where all X_j are RS of finite type. W.l.o.g. ${}^{W}\!/_{G} = X_1 \dot\cup \cdots \dot\cup X_k$ with some $k \le l$. Put $W_j := p^{-1}(X_j)$, the preimage of X_j. So $W = W_1 \dot\cup \cdots \dot\cup W_k$. Let V_j be one of the components of W_j, and $G_j \le G$ the stabilizer of V_j. Then $X_j = {}^{W_j}\!/_{G} = {}^{V_j}\!/_{G_j}$ and $W_j = \dot\bigcup_{gG_j \in {}^{G}/_{G_j}} g(V_j)$. It follows that the essential part of $\Lambda(G)$ in Proposition 96, p. 61, is $\partial W_j = \bigcup_{gG_j \in {}^{G}/_{G_j}} g(\partial V_j)$.

PROPOSITION 104:

$$M(G,W) = \prod_{j=1}^{k} M(G,W_j) = \prod_{j=1}^{k} M(G_j,V_j)$$

Note: G_j need not be finitely generated, but that doesn't matter because we already know that ${}^{V_j}\!/_{G_j}$ is of finite type.

COROLLARY 105:

$$M_{\mathrm{triv}}(G,W) = \prod_{j=1}^{k} M_{\mathrm{triv}}(G,W_j)$$

$$T(G,W) = \prod_{j=1}^{k} T(G,W_j)$$

COROLLARY 106: $M_{\mathrm{triv}}(G,W_j) = M_{\mathrm{triv}}(G_j,V_j)$, so $T(G,W_j) = T(G_j,V_j)$.

Example 98 (continued):
$T(G, \mathbb{C} \setminus \mathbb{R}) = T(G, \mathbb{H}) \times T(G, -\mathbb{H}) = T(G, \mathbb{H}) \times \overline{T(G, \mathbb{H})} = T_g \times \overline{T_g}$. (See also [Nag 88, Prop. 4.2.2., p. 270].)
Let us take a closer look at $T(G_j, V_j)$. V_j is a G-invariant component of $\Omega(G_j)$.

3.2.5. The single components.
Omitting the index j, let V be a G-invariant component of Ω, and $X = {}^{V}/_{G}$.
If V is simply connected,

$$M_{\text{triv}}(G, V) = \left\{ \begin{aligned} f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f \text{ qc.}, & \\ f|_{\widehat{\mathbb{C}} \setminus V} \text{ conformal}, & \\ f|_{\Lambda} = \mathrm{id}_{\Lambda}, & \\ f \circ g \circ f^{-1} = g \quad \forall g \in G & \end{aligned} \right\}$$
$$= \left\{ f \in M(G, V) : f|_{\overline{V}} \text{ homotopic to } \mathrm{id}_{\overline{V}} \mod \partial V \right\}$$

and $T(G, V)$ turns out to be the classical Teichmüller space $T(G, V) = T_{V/_G}$.
In general, let $\tilde{V}$ be the universal covering of V. Since $\tilde{V} \cong \mathbb{D}$, this is called a **Fuchsian model**.

$$\begin{tikzcd} \tilde{V} \arrow[d, "h"', "/H"] \arrow[dd, bend left=60, "/\Gamma"] \\ V \arrow[d, "p"', "/G"] \\ {}^{V}/_{G} = X \end{tikzcd}$$

Put $\Gamma := \mathcal{D}eck\left(\tilde{V}/X\right) \leq \mathrm{Aut}_{\mathbf{RS}}\left(\tilde{V}\right)$. There is a homomorphism $\chi : \Gamma \to G$ such that

$$\begin{tikzcd} \tilde{V} \arrow[r, "\gamma"] \arrow[d, "h"'] & \tilde{V} \arrow[d, "h"] \\ V \arrow[r, "\chi(\gamma)"'] & V \end{tikzcd}$$

commutes: $h \circ \gamma = \chi(\gamma) \circ h \ \forall \gamma \in \Gamma$.
Put $H := \mathrm{kern}\,(\chi) = \left\{ \gamma \in \mathrm{Aut}_{\mathbf{RS}}\left(\tilde{V}\right) : h \circ \gamma = h \quad \forall \gamma \in \Gamma \right\} = \mathcal{D}eck\left(\tilde{V}/V\right) \cong \pi_1(V)$.
Then ${}^{\Gamma}/_{H} \cong \mathrm{im}\,\chi = G$. Clearly, we have a norm preserving isomorphism

$$\left\{ \begin{aligned} M(G, V) \to M\left(\Gamma, \tilde{V}\right) & \\ \mu \mapsto \mu \circ h & \\ \mu \circ h^{-1} \hookleftarrow \mu & \end{aligned} \right\}$$

Consider pairs (F, θ) satisfying the following conditions:

(i) $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ qc, with $\mu_F \in M\left(\Gamma, \tilde{V}\right)$. Then we have a well-defined, injective homomorphism

$$\theta_F : \left\{ \begin{array}{l} \Gamma \to \mathrm{Aut}_{\mathbf{RS}}\left(\tilde{V}\right) \\ \gamma \mapsto F \circ \gamma \circ F^{-1} \end{array} \right\}$$

(ii) $\theta_F(H) = H$. Then F induces a qc homeomorphism $f : V \to V$ such that

$$\begin{tikzcd} \tilde{V} \arrow[r, "F"] \arrow[d, "h"'] & \tilde{V} \arrow[d, "h"] \\ V \arrow[r, "f"'] & V \end{tikzcd}$$

commutes.

(iii) $\theta_F(\Gamma) = \Gamma$. Then f comes from a qc homeomorphism $X \to X$, i.e. $fGf^{-1} = G$.

Definition 107: $\mathrm{Mod}^H\left(\Gamma, \tilde{V}\right) := \{(F, \theta) : \text{(i), (ii), (iii)}\}$.

With one more condition

(iv) $\forall g \in G : fgf^{-1} = g$, or (equivalently) $\forall \gamma \in \Gamma : F\gamma F^{-1}\gamma^{-1} \in H$

we define

Definition 108: $\mathrm{Mod}_H\left(\Gamma, \tilde{V}\right) := \{(F, \theta) : \text{(i), (ii), (iii), (iv)}\}$.

Fact 109: $\mathrm{Mod}_H\left(\Gamma, \tilde{V}\right) \trianglelefteq \mathrm{Mod}^H\left(\Gamma, \tilde{V}\right) \leq \mathrm{Mod}\left(\Gamma, \tilde{V}\right)$.

$\mathrm{Mod}_H\left(\Gamma, \tilde{V}\right)$ operates on $T\left(\Gamma, \tilde{V}\right)$, and we have

Theorem 110:

$$T(G, V) \cong T\left(\Gamma, \tilde{V}\right) / \mathrm{Mod}_H\left(\Gamma, \tilde{V}\right)$$

The operation of $\mathrm{Mod}_H\left(\Gamma, \tilde{V}\right)$ is freely discontinuous, therefore $T(G, V)$ is a complex manifold of the same dimension as $T\left(\Gamma, \tilde{V}\right)$.

3.2.6. Putting it together.

In the setting of section 3.2.4, we now have

Corollary 111:

$$T(G, W) = \prod_{j=1}^{k} T(G, W_j) = \prod_{j=1}^{k} T(G_j, V_j) = \prod_{j=1}^{k} T\left(\Gamma_j, \tilde{V}_j\right) / \mathrm{Mod}_{H_j}\left(\Gamma_j, \tilde{V}_j\right)$$

Definition 112:

$$\mathrm{Mod}'(G,W) := \{(F,\theta) \in \mathrm{Mod}(G,W) : F \text{ keeps every } W_j \subset W \text{ invariant}\}$$

Note 113: $\mathrm{Mod}'(G,W) \cong \prod_{j=1}^{k} \mathrm{Mod}^{H_j}\left(\Gamma_j, \tilde{V_j}\right) / \prod_{j=1}^{k} \mathrm{Mod}_{H_j}\left(\Gamma_j, \tilde{V_j}\right)$.

Note 114: $\mathrm{Mod}'(G,W) \trianglelefteq \mathrm{Mod}(G,W)$, a normal subgroup of finite index.

Corollary 115: $\mathcal{M}(G,W) = T(G,W)/\mathrm{Mod}(G,W)$ is the factor of

$$T(G,W)/\mathrm{Mod}'(G,W) = \prod_{j=1}^{k} \left(T\left(\Gamma_j, \tilde{V_j}\right) / \mathrm{Mod}^{H_j}\left(\Gamma_j, \tilde{V_j}\right)\right)$$

by a finite group, therefore $\mathcal{M}(G,W)$ is a normal complex space.

The last assertion is a consequence of a theorem of H. Cartan, see [Imayoshi, Taniguchi 92, § 6.3, p. 166-167].

3.3. The deformation space of G.
Let $G_{t,s} \cong G \leq \mathrm{PGL}_2(\mathbb{C})$ a Kleinian group, as in section 2.4, and $g \geq 1$.

Theorem 116: $T(G,\Omega)$ is a complex analytic manifold of (complex) dimension $3(g-1+r-k)$.

Proof: Recall that $G = G_1 * \cdots * G_k * \mathbb{Z}^{*s}$ with $\Delta_{t_j} \cong G_j \leq G \leq \mathrm{PGL}_2(\mathbb{C})$. From the proof of Theorem 67, p. 42, we have the following geometric description of Ω: There is a G-invariant component Ω', and for each $j \in \{1,\ldots,k\}$ with $t_j > 1$ there is a component Ω''_j. Ω''_j is simply connected, by [Maskit 88, IX.D.2., p. 221]. The decomposition of Ω/G is

$$\Omega/G = \Omega'/G \,\dot\cup\, \dot\bigcup_{t_j>1} \left(\Omega''_j/G_j\right)$$

The decomposition of Ω itself is

$$\Omega = \Omega' \,\dot\cup\, \dot\bigcup_{t_j>1} \dot\bigcup_{gG_j \in G/G_j} g\left(\Omega''_j\right)$$

Therefore

$$\begin{aligned} T(G,\Omega) &= T(G,\Omega') \times \prod_{t_j>1} T\left(G_j, \Omega''_j\right) \\ &= T\left(\Gamma, \widetilde{\Omega'}\right) / \mathrm{Mod}_H\left(\Gamma, \widetilde{\Omega'}\right) \times \prod_{t_j>1} T\left(G_j, \Omega''_j\right) \end{aligned}$$

On the right hand side, we have complex manifolds. We can sum up the dimensions:

$$\begin{aligned}
\dim_{\mathbb{C}} T(G,\Omega) &= \dim_{\mathbb{C}} T\left(\Gamma, \widetilde{\Omega'}\right) + \sum_{t_j>1} \dim_{\mathbb{C}} T\left(G_j, \Omega_j''\right) \\
&= (3g-3) + \sum_{t_j>1} (3t_j - 3) \\
&= 3\left(g - 1 + \sum_j (t_j - 1)\right) \\
&= 3(g - 1 + r - k)
\end{aligned}$$

q.e.d.

3.4. Global holomorphic coordinates for the deformation space of G.
In this section, we find global holomorphic coordinates for the deformation space of G, similar in spirit to the "Maskit coordinates" [Bers 81, § 13].
Notation: Let Ω denote the region of discontinuity and Λ the limit set of a Kleinian group G. Let $\mathrm{DV}\,(z_1, z_2, z_3, z_4) := \dfrac{z_1 - z_3}{z_2 - z_3} : \dfrac{z_1 - z_4}{z_2 - z_4}$ denote the **cross ratio** of four distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$. w^μ denotes the solution of the Beltrami equation, as in [Nag 88, p. 34].

<u>THEOREM 117:</u> Let $G_{t,s} \cong G \leq \mathrm{PGL}_2(\mathbb{C})$ a Kleinian group, as in section 2.4, and $g \geq 2$. Then there are p points $x_1, \ldots, x_p \in \Lambda$ and $4q$ points $x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, \ldots,$ $x_{q,1}, x_{q,2}, x_{q,3}, x_{q,4} \in \Lambda$ with $N := p + q = 3(g - 1 + r - k)$, such that

$$\Phi : \left\{ \begin{array}{lc} T(G,\Omega) \rightarrow & \mathbb{C}^N \\ \quad \mu \quad\ \mapsto & \begin{pmatrix} w^\mu(x_1) \\ \vdots \\ w^\mu(x_p) \\ \mathrm{DV}\,(w^\mu(x_{1,1}), w^\mu(x_{1,2}), w^\mu(x_{1,3}), w^\mu(x_{1,4})) \\ \vdots \\ \mathrm{DV}\,(w^\mu(x_{q,1}), w^\mu(x_{q,2}), w^\mu(x_{q,3}), w^\mu(x_{q,4})) \end{pmatrix} \end{array} \right\}$$

is a biholomorphic homeomorphism onto a region $\subset \mathbb{C}^N$.

This theorem is a generalization of [Nag 88, Theorem 4.2.3., p.270].
Proof: $G \cong G_{t,s}$ is generated by

(a) loxodromic generators $c \in \mathrm{PGL}_2(\mathbb{C})$ (s of them),
(b) parabolic generators $a, b \in \mathrm{PGL}_2(\mathbb{C})$ with $[a,b] = 1$ (once for each j with $t_j = 1$),
(c) loxodromic generators $a, b \in \mathrm{PGL}_2(\mathbb{C})$ with $[a,b] \neq 1$ (t_j pairs of this for each j with $t_j > 1$).

For each of these sets of generators, we choose appropriate points x_i or $x_{i,j}$.

(a) A loxodromic $c \in \mathrm{PGL}_2(\mathbb{C})$ is uniquely determined by

- $z_1 :=$ the attracting fixed point of c,
- $z_2 :=$ the repelling fixed point of c,
- $\lambda = \mathrm{DV}\,(c(z_3), z_3, z_1, z_2)$ for any $z_3 \in \widehat{\mathbb{C}} \setminus \{z_1, z_2\}$.

Proof: W.l.o.g. (conjugation in $\mathrm{PGL}_2(\mathbb{C})$) we may assume $z_1 = 0$, $z_2 = \infty$, i.e. $c = (z \mapsto \lambda z)$ with $|\lambda| < 1$. Indeed $\mathrm{DV}\,(c(z), z, z_1, z_2) = \mathrm{DV}\,(\lambda z, z, 0, \infty) = \frac{\lambda z}{z} = \lambda$.

(b) A pair (a, b) of commuting parabolic elements of $\mathrm{PGL}_2(\mathbb{C})$ is uniquely determined by

- $z_1 :=$ common fixed point of a and b.
- $\tau = \mathrm{DV}\,(a(z_4), b(z_4), z_4, z_1)$ for any $z_4 \in \widehat{\mathbb{C}} \setminus \{z_1\}$,
- $\sigma = \mathrm{DV}\,(a(z_2), z_2, z_3, z_1)$ where z_2, z_3 are some other points.

Proof: W.l.o.g. (conjugation in $\mathrm{PGL}_2(\mathbb{C})$) we may assume $z_1 = \infty$, i.e. $a = (z \mapsto z + \alpha)$, $b = (z \mapsto z + \beta)$ with $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Indeed, $\sigma = \mathrm{DV}\,(a(z_2), z_2, z_3, z_1) = \mathrm{DV}\,(z_2 + \alpha, z_2, z_3, \infty) = 1 + \frac{\alpha}{z_2 - z_3}$ determines α, and $\tau = \mathrm{DV}\,(a(z_4), b(z_4), z_4, z_1) = \mathrm{DV}\,(z_4 + \alpha, z_4 + \beta, z_4, \infty) = \frac{\alpha}{\beta}$ determines β.

(c) A pair (a, b) of non-commuting loxodromic elements of $\mathrm{PGL}_2(\mathbb{C})$ is uniquely determined by: Put $c := [a, b] = aba^{-1}b^{-1} \neq 1$.

- $z_1 :=$ the attracting fixed point of c,
- $z_2 :=$ the repelling fixed point of c,
- $z_3 := a(z_1)$,
- $z_4 := b(z_1)$,
- $z_5 := a(z_2)$,
- $z_6 := c(z_4) = aba^{-1}(z_1)$.

(Cf. [Kra, Maskit 81, p. 1077-1079], but the six points we use are slightly different from theirs.)

Proof: W.l.o.g. $z_1 = 0$, $z_2 = \infty$. W.l.o.g. $z_4 = 1$. Then $c : \left\{ \begin{array}{c} 0 \mapsto 0 \\ \infty \mapsto \infty \\ 1 \mapsto z_6 \end{array} \right\}$ has the form $c \,\widehat{=} \begin{pmatrix} z_6 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{C})$. Put $\lambda = \pm\sqrt{z_6}$. Then c has the form $c \,\widehat{=} \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ in $\mathrm{PSL}_2(\mathbb{C})$. This means $c = \left(z \mapsto \lambda^2 z\right)$, and since 0 is its attracting fixed point, we have $0 < |\lambda| < 1$. Write $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $b = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$, both $\in \mathrm{SL}_2(\mathbb{C})$. In $\mathrm{PSL}_2(\mathbb{C})$ we have the equation $c = aba^{-1}b^{-1}$. Choose the sign of λ in such a way that this equation still holds in $\mathrm{SL}_2(\mathbb{C})$. We then have

$$\begin{aligned} \alpha' + \delta' = \mathrm{Tr}\, b = \mathrm{Tr}\left(aba^{-1}\right) = \mathrm{Tr}\,(cb) = \mathrm{Tr} \begin{pmatrix} \lambda\alpha' & \lambda\beta' \\ \frac{1}{\lambda}\gamma' & \frac{1}{\lambda}\delta' \end{pmatrix} \\ = \lambda\alpha' + \frac{1}{\lambda}\delta' \end{aligned}$$

$\Rightarrow (1 - \lambda)\left(\alpha' - \frac{1}{\lambda}\delta'\right) = 0 \Rightarrow \alpha' = \frac{1}{\lambda}\delta'$. Of course also $1 = z_4 = b(z_1) = b(0) = \frac{\beta'}{\delta'}$, so $\beta' = \delta'$.

$$b = \begin{pmatrix} \frac{1}{\lambda}\delta' & \delta' \\ \gamma' & \delta' \end{pmatrix}$$

Furthermore

$$\alpha + \delta = \operatorname{Tr} a^{-1} = \operatorname{Tr}\left(ba^{-1}b^{-1}\right) = \operatorname{Tr}\left(a^{-1}c\right) = \operatorname{Tr}\begin{pmatrix} \lambda\delta & -\frac{1}{\lambda}\beta \\ -\lambda\gamma & \frac{1}{\lambda}\alpha \end{pmatrix}$$
$$= \lambda\delta + \frac{1}{\lambda}\alpha$$

$\Rightarrow\ (1-\lambda)\big(\delta - \frac{1}{\lambda}\alpha\big) = 0\ \Rightarrow\ \alpha = \lambda\delta$. So $a = \begin{pmatrix} \lambda\delta & \beta \\ \gamma & \delta \end{pmatrix}$. From $z_3 = a(z_1) = a(0) = \frac{\beta}{\delta}$ and $z_5 = a(z_2) = a(\infty) = \frac{\lambda\delta}{\gamma}$ we get $\beta = z_3\delta$, $\gamma = \frac{\lambda\delta}{z_5}$, so

$$a = \begin{pmatrix} \lambda\delta & z_3\delta \\ \frac{\lambda}{z_5}\delta & \delta \end{pmatrix}$$

Since this is the representation of a in $\mathrm{SL}_2(\mathbb{C})$, we know that $1 = \det a = \lambda\Big(1 - \frac{z_3}{z_5}\Big)\delta^2$, This determines δ^2, so a is uniquely determined in $\mathrm{PSL}_2(\mathbb{C})$. Now we have in $\mathrm{SL}_2(\mathbb{C})$ the equation $cba = ab$

$$\Leftrightarrow \quad \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda}\delta' & \delta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} \lambda\delta & z_3\delta \\ \frac{\lambda}{z_5}\delta & \delta \end{pmatrix} = \begin{pmatrix} \lambda\delta & z_3\delta \\ \frac{\lambda}{z_5}\delta & \delta \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda}\delta' & \delta' \\ \gamma' & \delta' \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} \lambda\Big(1+\frac{\lambda}{z_5}\Big)\delta'\delta & (z_3+\lambda)\delta'\delta \\ \Big(\gamma' + \frac{\delta'}{z_5}\Big)\delta & \frac{z_3\gamma'+\delta'}{\lambda}\delta \end{pmatrix} = \begin{pmatrix} \delta\delta' + z_3\gamma'\delta & (z_3+\lambda)\delta\delta' \\ \frac{1}{z_5}\delta\delta' + \gamma'\delta & \Big(\frac{\lambda}{z_5}+1\Big)\delta\delta' \end{pmatrix}$$

This and $1 = \det b = \big(\frac{1}{\lambda}\delta' - \gamma'\big)\delta'$ leads to

$$\gamma' = \frac{1}{\lambda}\delta' - \frac{1}{\delta'}$$
$$\lambda\Big(1+\frac{\lambda}{z_5}\Big) = 1 + z_3\frac{\gamma'}{\delta'}$$

These two equations imply

$$\lambda\Big(1+\frac{\lambda}{z_5}\Big) = 1 + z_3\Big(\frac{1}{\lambda} - \frac{1}{\delta'^2}\Big)$$

This determines δ'^2 uniquely, and $\frac{\gamma'}{\delta'}$ as well, so b can be uniquely computed in $\mathrm{PSL}_2(\mathbb{C})$:

$$a = \delta\cdot\begin{pmatrix} \lambda & z_3 \\ \frac{\lambda}{z_5} & 1 \end{pmatrix}, \quad b = \delta'\cdot\begin{pmatrix} \frac{1}{\lambda} & 1 \\ \frac{1}{z_3}\Big(\lambda\Big(1+\frac{\lambda}{z_5}\Big)-1\Big) & 1 \end{pmatrix}$$

Let us now choose p, q and the set of $N = p + q$ coordinates. Assume the notation of Formula 16, p. 23. Let $C := \dot{\bigcup}_{j=1}^{k}\dot{\bigcup}_{i=1}^{t_j}\{a_{j,i}, b_{j,i}\} \dot{\cup} \dot{\bigcup}_{j=1}^{s}\{c_j\}$, a set of generators of $G_{t,s}$, as an ordered set.

- ⋆ Case 1: $s > 0$ or some $t_j > 1$.
  - (a) For every $j \in \{1, \ldots, s\}$, handle the generator c_j: As described above, choose some z_1, z_2, and λ belonging to z_3, where z_3 is the attracting fixed point of the first loxodromic element in $C \setminus \{t_j\}$.
  - (b) For every $j \in \{1, \ldots, k\}$ with $t_j = 1$, handle the generators $a_{j,1}$ and $b_{j,1}$: As described above, choose z_1, τ, σ, where $z_2 = z_4$ and z_3 are the attracting and repelling fixed points of the first loxodromic element in C, respectively.
  - (c) For every $j \in \{1, \ldots, k\}$ with $t_j > 1$, handle the generators $a_{j,i}, b_{j,i}$ ($i \in \{1, \ldots, t_j - 1\}$) by choosing the six points, as described above, and the generators $a_{j,i}, b_{j,i}$ ($i = t_j$) by choosing only the three points z_3, z_4, z_5. The points z_1, z_2, z_6 for $i = t_j$ are redundant since the commutator $\left[a_{j,t_j}, b_{j,t_j}\right] = \left(\prod_{i=1}^{t_j-1} [a_{j,i}, b_{j,i}]\right)^{-1}$ will already be known from the previous $6(t_j - 1)$ points.
- ⋆ Case 2: $s = 0$ and all $t_j = 1$, but $k > 2$.
  - (b) For every $j \in \{1, \ldots, k\}$, handle the generators $a_{j,1}$ and $b_{j,1}$: As described above, choose z_1, τ, σ, where $z_2 = z_4$ and z_3 are the common fixed points of the first two pairs of generators in $C \setminus \{a_{j,1}, b_{j,1}\}$.

This gives

$$\begin{aligned} p &= 2 \cdot s + 1 \cdot \#\{j : t_j = 1\} + \sum_{t_j > 1} (6t_j - 3) \\ &= 2s + \sum_j (6t_j - 3) - 2 \cdot \#\{j : t_j = 1\} \\ &= 2s + 6r - 3k - 2 \cdot \#\{j : t_j = 1\} \end{aligned}$$

points and $q = 1 \cdot s + 2 \cdot \#\{j : t_j = 1\}$ cross-ratios, so $N = p + q = 3s + 6r - 3k = 3(g + r - k)$.

- ⋆ Case 3: $s = 0$, $k = 2$, $t_1 = t_2 = 1$, i.e. $G = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] = [a_2, b_2] = 1 \rangle$. The above procedure would give us only $p = 2$ points. In order to get at least three points, put $c := [a_1, a_2]$, a loxodromic element, and choose
  - $z_1 :=$ the common fixed point of a_1 and b_1,
  - $z_2 :=$ the common fixed point of a_2 and b_2,
  - $z_3 :=$ the attracting fixed point of c,
  - $\tau_1 = \mathrm{DV}\,(a_1(z_3), b_1(z_3), z_3, z_1)$,
  - $\tau_2 = \mathrm{DV}\,(a_2(z_3), b_2(z_3), z_3, z_2)$,
  - $\sigma_1 = \mathrm{DV}\,(a_1(z_2), z_2, z_3, z_1)$.

  These determine a_1, b_1, a_2, b_2 uniquely.
  Proof: W.l.o.g. $z_1 = 0$, $z_2 = \infty$. We already saw that z_1, τ_1, σ_1 determine a_1 and b_1 uniquely. Now write (in $\mathrm{SL}_2(\mathbb{C})$)

$$a_1 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C} \setminus \{0\}$. We have

$$\begin{aligned} c &= a_1 a_2 {a_1}^{-1} {a_2}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\alpha\gamma & \alpha\gamma^2 \\ -\alpha^2\gamma & 1+\alpha\gamma+\alpha^2\gamma^2 \end{pmatrix} \end{aligned}$$

From $c(z_3) = z_3$ we get $\dfrac{(1-\alpha\gamma)\cdot z_3 + \alpha\gamma^2}{-\alpha^2\gamma \cdot z_3 + (1+\alpha\gamma+\alpha^2\gamma^2)} = z_3 \Leftrightarrow \alpha {z_3}^2 - (2+\alpha\gamma) z_3 + \gamma = 0$, so this determines $\gamma = \dfrac{\alpha {z_3}^2 - 2z_3}{\alpha z_3 - 1}$. $\tau_2 = \dfrac{\gamma}{\delta}$ now determines δ.

In this case, $p = 3$, $q = 3$, so $N = p + q = 3(g + r - k)$ as well.

We thus have associated to G: $p \geq 3$ points and q cross-ratios. All points involved are fixed points of some specific elements of G, therefore in Λ.

We now apply the above construction to G^μ (as defined in Corollary 90, p. 60) instead of G. Instead of a point x_i, we now take $w^\mu(x_i)$, which is the corresponding fixed point of the corresponding element of G^μ.

Finally, w^μ can be normalized and made unique by the constraints $w^\mu(x_1) = 0$, $w^\mu(x_2) = 1$, $w^\mu(x_3) = \infty$. This amounts to replacing $w^\mu(z)$ by $\mathrm{DV}\,(w^\mu(z), w^\mu(x_2), w^\mu(x_1), w^\mu(x_3))$. (This will reduce p and N by 3, at the end.)

We can now actually write down Φ:

$$\Phi : \left\{ \begin{array}{ccc} T(G,\Omega) & \to & \mathbb{C}^{N-3} \\ \mu & \mapsto & \begin{pmatrix} \mathrm{DV}\,(w^\mu(x_4), w^\mu(x_2), w^\mu(x_1), w^\mu(x_3)) \\ \vdots \\ \mathrm{DV}\,(w^\mu(x_p), w^\mu(x_2), w^\mu(x_1), w^\mu(x_3)) \\ \mathrm{DV}\,(w^\mu(x_{1,1}), w^\mu(x_{1,2}), w^\mu(x_{1,3}), w^\mu(x_{1,4})) \\ \vdots \\ \mathrm{DV}\,(w^\mu(x_{q,1}), w^\mu(x_{q,2}), w^\mu(x_{q,3}), w^\mu(x_{q,4})) \end{pmatrix} \end{array} \right\}$$

Claim 1: Φ is well-defined.

Proof: If $\mu_1, \mu_2 \in M(G,\Omega)$ with $\mu_1 = \mu_2$ in $T(G,\Omega)$, there is some $f \in M_{\mathrm{triv}}(G,\Omega)$ such that $\mu_2 = \mu_1 \circ f$ in $M(G,\Omega)$, so $h \circ w^{\mu_2} = w^{\mu_1} \circ f$ with some $h \in \mathrm{PGL}_2(\mathbb{C})$. $f \in M_{\mathrm{triv}}(G,\Omega)$ implies $\forall g \in G : f \circ g \circ f^{-1} = g$, so $h \circ g^{\mu_2} \circ h^{-1} = h \circ w^{\mu_2} \circ g \circ (w^{\mu_2})^{-1} \circ h^{-1} = w^{\mu_1} \circ f \circ g \circ f^{-1} \circ (w^{\mu_1})^{-1} = w^{\mu_1} \circ g \circ (w^{\mu_1})^{-1} = g^{\mu_1}$. The normalization of w^μ implies that $h = \mathrm{id}$, so $g^{\mu_2} = g^{\mu_1}$ $\forall g \in G$. Since the points x_j and $x_{j,i}$ depend on the group elements only (by definition), we see that $\Phi(\mu_2) = \Phi(\mu_1)$.

Claim 2: Φ is injective.

Proof: The lengthy computations above show that if $\Phi(\mu_1) = \Phi(\mu_2)$ then the corresponding generators of G^{μ_1} and G^{μ_2} are equal. In particular, $G^{\mu_1} = G^{\mu_2}$.

Consider $f := w^{\mu_2} \circ (w^{\mu_1})^{-1} \in \mathrm{Aut}_{\mathrm{qc}}\left(\widehat{\mathbb{C}}\right)$. f is a qc homeomorphism $\Omega^{\mu_1} \to \Omega^{\mu_2}$. $f \in M(G^{\mu_1}, \Omega^{\mu_1})$.

For every loxodromic $g \in G$, $g^{\mu_1} = w^{\mu_1} \circ g \circ (w^{\mu_1})^{-1}$ and $g^{\mu_2} = w^{\mu_2} \circ g \circ (w^{\mu_2})^{-1}$ are loxodromic as well. Let z_i ($i \in \{1,2\}$) be the attracting fixed point of g^{μ_i}. Since $f \circ g^{\mu_1} \circ$

$f^{-1} = g^{\mu_2}$, f maps z_1 to z_2. On the other hand, since corresponding generators of G^{μ_1} and G^{μ_2} are equal, we have $g^{\mu_1} = g^{\mu_2}$, so $z_1 = z_2$. Thus $f(z_1) = z_2 = z_1$. Similarly, $f(z) = z$ if z is the fixed point of some parabolic $g \in G^{\mu_1}$. Since the limit set Λ^{μ_1} of G^{μ_1} is the closure of the set of fixed points of loxodromic and parabolic elements of G^{μ_1} (see [Bers 72, § 1.3., p. 261-262]), we have $f(z) = z$ for all $z \in \Lambda^{\mu_1}$. Therefore $f \in M_{\text{triv}}(G^{\mu_1}, \Omega^{\mu_1})$, $(w^{\mu_1})^{-1} \circ w^{\mu_2} = (w^{\mu_1})^{-1} \circ f \circ w^{\mu_1} \in M_{\text{triv}}(G, \Omega)$, so $\mu_1 = \mu_2$ in $T(G, \Omega)$.

CLAIM 3: Φ is holomorphic.
This follows from the holomorphy of the mapping $\mu \mapsto w^\mu$, see [Nag 88, § 1.2.11., p. 38].
By the open mapping theorem (Brouwer for holomorphic mappings) Φ is a biholomorphic homeomorphism onto its image. By Theorem 116, p. 65, $T(G, \Omega)$ is an open, connected set of complex dimension $3(g - 1 + r - k)$, so im Φ is an open, connected set in $\mathbb{C}^{N-3}$. **q.e.d.**

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Lebenslauf

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