D-finite Power Series and the Diagonal Theorem

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This paper gives an introduction into *differentiably finite* power series and presents a proof and an application of the diagonal theorem.

1. Definitions

DEFINITION 1:
An *X-differential ring* R is an integral domain R of characteristic zero, together with a finite set $X = \{\partial_1, \ldots, \partial_d\}$ of pairwise commuting derivations ∂_i on R.

See [Hai], section 2.1.

DEFINITION 2:
Let R be an X-differential ring. An *X-differential R-module* M is an R-module M, together with a rule how the derivations of X operate on M: For every $\partial \in X$ there is a map $\partial_M : M \to M$ (which we denote by the same symbol ∂) such that $\partial(f+g) = \partial(f) + \partial(g)$ for every $f, g \in M$ and $\partial(af) = \partial(a) \cdot f + a \cdot \partial(f)$ for every $a \in R$ and $f \in M$, and these maps $\partial \in \mathrm{Der}_R(M)$ must be pairwise commuting.

See [Hai], section 2.2.
The definitions 1 and 2 can also be found in [Ko], p. 66.

DEFINITION 3:
Let R be an X-differential ring. An *X-differential R-algebra* M is an X-differential R-module M together with a (not necessarily commutative) multiplication on M such that $(M, \cdot)$ is an R-algebra and for every $\partial \in X$, $f, g \in M$ we have $\partial(f \cdot g) = \partial(f) \cdot g + f \cdot \partial(g)$.

See [Hai], section 2.3.

PROPOSITION 1:

(a) If R is an X-differential ring, then $R[z]$ is an $(X \cup \{\partial_z\})$-differential ring, where ∂_z denotes the usual partial derivative with respect to z, uniquely determined by $\partial_z|_R = 0$ and $\partial_z(z) = 1$.
(b) If M is an X-differential R-module, then $M[\![z]\!]$ is an $(X \cup \{\partial_z\})$-differential $R[z]$-module, where the operation of ∂_z on $M[\![z]\!]$ is defined by $\partial_z\left(\sum_{i=0}^{\infty} f_i z^i\right) = \sum_{i=1}^{\infty} i f_i z^{i-1}$. Furthermore, if M is a torsion-free R-module, then $M[\![z]\!]$ is a torsion-free $R[z]$-module.

Proof: See [Hai], section 2.5.

PROPOSITION 2:

(a) If R is an X-differential ring, then the quotient field $\mathrm{Quot}(R)$ is an X-differential ring.
(b) If M is an X-differential R-module, then $\widetilde{M} := (R \setminus \{0\})^{-1} M = \mathrm{Quot}(R) \underset{R}{\otimes} M$ is an X-differential $\mathrm{Quot}(R)$-module, especially a $\mathrm{Quot}(R)$ vector space.
(c) If M is an X-differential R-algebra, then $\widetilde{M}$, defined as in (b), is an X-differential $\mathrm{Quot}(R)$-algebra.

Proof: See [Hai], sections 2.6, 2.7, 2.8.

2. D-finiteness in general

DEFINITION 4:
Let M be an X-differential R-module. Put $\widetilde{R} := \mathrm{Quot}\,(R)$ and $\widetilde{M} := (R \setminus \{0\})^{-1} M$; then $\widetilde{M}$ is an X-differential $\widetilde{R}$-module as in Prop. 2. Let $\lambda: M \to \widetilde{M}$ be the canonical map. For $f \in M$ let $\widetilde{f} := \lambda(f) \in \widetilde{M}$ be its image and $A(f) := \sum_{n \in \mathbb{N}^d} \widetilde{R} \cdot \partial^n \widetilde{f}$ (with $\partial^n \widetilde{f}$ meaning ${\partial_1}^{n_1} \cdots {\partial_d}^{n_d} \widetilde{f}$) the span of all its derivatives, a subspace of the $\widetilde{R}$-vector space $\widetilde{M}$. f is called *differentiably finite* – or *D-finite* for short –, and we then write $f \in D(R, M, X)$, if $\dim_{\widetilde{R}} A(f) < \infty$.

See [Hai], section 2.9.

PROPOSITION 3:
Let M be an X-differential R-module. Then

(a) $0 \in D(R, M, X)$.
(b) If $f \in D(R, M, X)$, then $-f \in D(R, M, X)$.
(c) If $f, g \in D(R, M, X)$, then $f + g \in D(R, M, X)$.
(d) If $f \in D(R, M, X)$ and $a \in R$, then $a \cdot f \in D(R, M, X)$.
(e) If M is an X-differential R-algebra and $f, g \in D(R, M, X)$, then $f \cdot g \in D(R, M, X)$.
(f) If $R \subset M$, then $R \subset D(R, M, X) \subset M$.
(g) $D(R, M, X)$ is an X-differential R-module.
Furthermore, if M is an X-differential R-algebra, then $D(R, M, X)$ is, too.

Proof:
(See [Hai], section 2.10.)

(a) $\dim A(0) = 0$.
(b) $A(-f) = A(f)$.
(c) $A(f + g) \subset A(f) + A(g)$.
(d) $A(a \cdot f) \subset A(f)$.
(e) $A(f \cdot g) \subset A(f) \cdot A(g)$ by the LEIBNIZ rule, so
$\dim A(f \cdot g) \leq \dim A(f) \cdot A(g) \leq \dim A(f) \underset{\widetilde{R}}{\otimes} A(g) = \dim A(f) \cdot \dim A(g)$.
(f) For $a \in R$, we have $\dim A(a) \leq 1$.
(g) Follows from (a) to (e) and the fact that $D(R, M, X)$ is closed w.r.t. taking derivatives because $A(\partial_i f) \subset A(f)$.

q.e.d.

PROPOSITION 4:
Let M be a commutative X-differential R-algebra with $1 \in M$, and $f \in M$.

(a) If f is algebraic over R, then $f \in D(R, M, X)$.
(b) If for every $i \in \{1, \ldots, d\}$ there is an element $g_i \in M$ algebraic over R such that $\partial_i f = g_i f$, then $f \in D(R, M, X)$.

Proof:
Denote by $\left\langle \widetilde{f} \right\rangle$ the subalgebra of $\widetilde{M}$ generated by $\widetilde{f}$, i.e. the $\widetilde{R}$-vector space generated by $1, \widetilde{f}, \widetilde{f}^2, \ldots$.

(a) $N := \left\langle \widetilde{f} \right\rangle$ is a field, a finite extension of $\widetilde{R}$. We have an equation $P\left(\widetilde{f}\right) = 0$ with a polynomial $P \in \widetilde{R}[T]$ of minimum degree. For $i \in \{1, \ldots, d\}$ we then have $0 = \partial_i \left(P\left(\widetilde{f}\right)\right) = P'\left(\widetilde{f}\right) \cdot \partial_i \widetilde{f} + (\partial_i P)\left(\widetilde{f}\right)$. Because of characteristic zero $P'\left(\widetilde{f}\right) \neq 0$, and it follows that $\partial_i \widetilde{f} \in N$. Thus $A(f) \subset N$, and $\dim_{\widetilde{R}} A(f) \leq \dim_{\widetilde{R}} N < \infty$.

(b) Put $N := \langle \widetilde{g_1}, \ldots, \widetilde{g_d} \rangle$. Then $A(f) \subset N f$, and it follows $\dim_{\widetilde{R}} A(f) \leq \dim_{\widetilde{R}} N \leq \prod_{i=1}^{d} \deg \widetilde{g_i} < \infty$.

q.e.d.

Proposition 5:

(a) D-finiteness w.r.t. to X is equivalent to simultaneous D-finiteness w.r.t. every single derivation in X:

$$D(R,M,X)=\bigcap_{i=1}^{d} D(R,M,\{\partial_i\})$$

(b) For a single derivation ∂:

$$D(R,M,\{\partial\})=\{f\in M : \text{there is a linear differential operator } L\in R[\partial]\setminus\{0\} \text{ with } Lf=0\}$$

Proof:

(b) Trivial; see [Hai], section 2.9.

(a) (See [Hai], section 2.11.) Let $f\in D(R,M,X)$. Then for any i, $A_{\{\partial_i\}}(f)\subset A_X(f)$, which is finite-dimensional, so $f\in D(R,M,\{\partial_i\})$. On the other hand, let $f\in\bigcap_{i=1}^{d}D(R,M,\{\partial_i\})$. For $i\in\{1,\dots,d\}$ find a nonzero differential operator $L_i\in R[\partial_i]$ with $L_i f=0$. Put $n_i:=\max(\deg L_i,1)$. Then

$$A(f)=\sum_{\substack{0\le m_1<n_1\\ \cdots\\ 0\le m_d<n_d}} \widetilde{R}\cdot{\partial_1}^{m_1}\cdots{\partial_d}^{m_d}f$$

and so $\dim A(f)\le n_1\cdots n_d<\infty$.

q.e.d.

3. D-finiteness over polynomial rings

We now specialize in the situation where R is a polynomial ring. Let G, the ground ring, be a Y-differential ring. Then the polynomial ring $R:=G[z_1,\dots,z_d]$ is an X-differential ring with $X=Y\cup Z$ and $Z=\{\partial_{z_1},\dots,\partial_{z_d}\}$, by Prop. 1(a). We will write $Z=\{\partial_1,\dots,\partial_d\}$.

Definition 5:

The *ring of differential operators with polynomial coefficients* $A:=G[z_1,\dots,z_d,Y,Z]$ is the free non-commutative $G[Y]$-algebra with the generators $z_1,\dots,z_d,\partial_1,\dots,\partial_d$ modulo the relations

$$z_iz_j=z_jz_i\quad\forall i,j$$

$$\partial_i\partial_j=\partial_j\partial_i\quad\forall i,j$$

$$\partial_iz_j=z_j\partial_i+\delta_{i,j}\quad\forall i,j,$$

and $G[Y]$ is a non-commutative ring ($\partial a=a\partial+\partial(a)$ for $a\in G$ and $\partial\in Y$). A is called the d-th Weyl *algebra* over $G[Y]$.

Some notation: We will enumerate Y as $Y=\{\partial_{-1},\dots,\partial_{-e}\}$. For $\alpha\in\mathbb{N}^d$ we write $z^\alpha={z_1}^{\alpha_1}\cdots{z_d}^{\alpha_d}$ and $|\alpha|=\alpha_1+\cdots+\alpha_d$. For $(\beta,\gamma)\in\mathbb{N}^d\times\mathbb{N}^e$ we write $\partial^{(\beta,\gamma)}={\partial_Z}^{\beta}{\partial_Y}^{\gamma}={\partial_1}^{\beta_1}\cdots{\partial_d}^{\beta_d}{\partial_{-1}}^{\gamma_1}\cdots{\partial_{-e}}^{\gamma_e}$.

We have

$$A=\bigoplus_{\substack{\alpha\in\mathbb{N}^d\\ \beta\in\mathbb{N}^{d+e}}} G\cdot z^\alpha\partial^\beta$$

Definition 6:

A *filtration* of a non-commutative G-algebra A is a sequence (A_n) of finite-dimensional G-modules A_n such that $A=\bigcup_{n=0}^{\infty}A_n$, $A_n\subset A_{n+1}$ for every n and $A_n\cdot A_m\subset A_{n+m}$ for every n,m. Here and henceforth we define the dimension of a G-module V by $\dim_G V:=\dim_{\mathrm{Quot}\,(G)}(G\setminus\{0\})^{-1}V=\dim_{\mathrm{Quot}\,(G)}\mathrm{Quot}\,(G)\underset{G}{\otimes}V$.

The WEYL algebra A has the filtration (A_n), with

$$A_n = \bigoplus_{\substack{\alpha\in\mathbb{N}^d \\ \beta\in\mathbb{N}^{d+e} \\ |\alpha|+|\beta|\leq n}} G\cdot z^\alpha\partial^\beta$$

This is because the multiplication on A is defined in such a way that

$$\partial^\beta z^\alpha = \prod_{i=1}^{d}\left(\sum_{\nu\geq 0}\nu!\binom{\alpha_i}{\nu}\binom{\beta_i}{\nu}z_i{}^{\alpha_i-\nu}\partial_i{}^{\beta_i-\nu}\right)\cdot\prod_{i=-e}^{-1}\partial_i{}^{\beta_i}$$

Let now M be an X-differential R-module. We then define an operation of A on M by putting

$$\left(\sum_{\alpha,\beta}c_{\alpha,\beta}z^\alpha\partial^\beta\right)\cdot f := \sum_{\alpha,\beta}c_{\alpha,\beta}\cdot z^\alpha\cdot\partial^\beta(f)$$

for $c_{\alpha,\beta}\in R$ ($\alpha\in\mathbb{N}^d,\beta\in\mathbb{N}^{d+e}$) and $f\in M$.

LEMMA 6:
Let $f\in M$. If there is a subalgebra A' of A such that

$$\dim_G(A'\cap A_n)\geq c\cdot n^{d+1}\quad\text{for } n\gg 0 \text{ and some } c>0$$

but

$$\dim_G(A'\cap A_n)f = O\big(n^d\big)\quad\text{for } n\gg 0,$$

then there is a differential operator $L\in A'\setminus\{0\}$ with $Lf=0$.

Proof:
The map

$$\left\{\begin{matrix} A'\cap A_n \to (A'\cap A_n)f \\ L \quad\mapsto\quad Lf\end{matrix}\right\}$$

is G-linear. For n sufficiently large it can't be injective because we have $\dim_G(A'\cap A_n)f < \dim_G(A'\cap A_n)$.

q.e.d.

In the following, we will also make use of the standard filtration of $R=G[z_1,\ldots,z_d]$ itself: $R=\bigcup_{n=0}^{\infty}R_n$ with

$$R_n = \bigoplus_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\leq n}} G\cdot z^\alpha = \big\{h\in R : \deg_{z_1,\ldots,z_d}h\leq n\big\}.$$

LEMMA 7:
If $f\in D(R,M,X)$, then there exists an $r\in\mathbb{N}$, an $s\in R_r\setminus\{0\}$ and a finite set $J\subset\mathbb{N}^{d+e}$ such that for every $\beta\in\mathbb{N}^{d+e}$:

$$s^{|\beta|}\cdot\partial^\beta f\in\sum_{\alpha\in J}R_{r|\beta|}\cdot\partial^\alpha f$$

Proof:
(This is a generalization of [Hai], 2.15.) The assumption is equivalent to $\dim_{\widetilde{R}}A(f)<\infty$. So a finite set of derivatives of $\widetilde{f}$ suffices to generate $A(f)$:

$$A(f) = \sum_{\alpha\in J}\widetilde{R}\cdot\partial^\alpha\widetilde{f}$$

with J finite. We may assume $0 \in J$ (otherwise extend J). Put $t := 1 + \max_{\alpha \in J} |\alpha|$. For every $\beta \in \mathbb{N}^{d+e}$ with $|\beta| \le t$ there is an equation

$$\partial^\beta \widetilde{f} = \sum_{\alpha \in J} \widetilde{c}_{\beta,\alpha} \partial^\alpha \widetilde{f}$$

with $\widetilde{c}_{\beta,\alpha} \in \widetilde{R}$ for every β and α. Considering common denominators, we can bring every such equation to the form

$$c_\beta \cdot \partial^\beta f = \sum_{\alpha \in J} (c_\beta \widetilde{c}_{\beta,\alpha}) \partial^\alpha f$$

with $c_\beta \in R \setminus \{0\}$ and $c_\beta \widetilde{c}_{\beta,\alpha} \in R$ for every α. Let s be a common nonzero multiple of all the c_β. Then for every β with $|\beta| \le t$ we have an equation

$$s \cdot \partial^\beta f = \sum_{\alpha \in J} c_{\beta,\alpha} \partial^\alpha f$$

with $c_{\beta,\alpha} \in R$ for every α. Put $r := \max(\{\deg s\} \cup \{\deg c_{\beta,\alpha} : \alpha \in J, |\beta| \le t\})$. We now prove the claim by induction on $|\beta|$. The case $\beta = 0$ is clear because of $0 \in J$. Now the induction step. Assume the claim for ∂^β:

$$s^{|\beta|} \cdot \partial^\beta f = \sum_{\alpha \in J} h_{\beta,\alpha} \cdot \partial^\alpha f \quad \text{with } h_{\beta,\alpha} \in R_{r|\beta|}$$

and apply any fixed $\partial_0 \in X$:

$$s^{|\beta|} \cdot \partial_0 \partial^\beta f + |\beta| \partial_0(s) s^{|\beta|-1} \cdot \partial^\beta f = \partial_0\left(s^{|\beta|} \cdot \partial^\beta f\right) = \sum_{\alpha \in J} \partial_0(h_{\beta,\alpha} \cdot \partial^\alpha f)$$

We have $\partial_0(R_n) \subset R_n$ (this is clear if $\partial_0 \in Y$ because then ∂_0 operates separately on every coefficient, and for $\partial_0 \in Z$ we even have $\partial_0(R_n) \subset R_{n-1}$). This implies $\deg \partial_0(s) \le \deg s \le r$. We thus get (after multiplication by s)

$$\begin{aligned}
s^{|\beta|+1} \cdot \partial_0 \partial^\beta f &= -|\beta| \partial_0(s) s^{|\beta|} \cdot \partial^\beta f + \sum_{\alpha \in J} s \partial_0(h_{\beta,\alpha}) \partial^\alpha f + \sum_{\alpha \in J} s h_{\beta,\alpha} \partial_0 \partial^\alpha f \\
&= \sum_{\alpha \in J} (-|\beta| \partial_0(s) h_{\beta,\alpha} + s \partial_0(h_{\beta,\alpha})) \partial^\alpha f + \sum_{\alpha \in J} h_{\beta,\alpha} \cdot s \partial_0 \partial^\alpha f \\
&\in \sum_{\alpha \in J} R_{r + r|\beta|} \partial^\alpha f + \sum_{\alpha \in J} R_{r|\beta|} \left(\sum_{\gamma \in J} R_r \partial^\gamma f \right) \\
&\subset \sum_{\alpha \in J} R_{r(1+|\beta|)} \partial^\alpha f
\end{aligned}$$

This proves the claim for $\partial_0 \partial^\beta$. **q.e.d.**

THEOREM 8:
Let M be a torsion-free X-differential R-module. Then $f \in D(R, M, X)$ if and only if $\dim_G A_n f = O(n^d)$ for $n \gg 0$.

Proof:

"$\Rightarrow$" Invoking Lemma 7, we get r, s and J. Since the map $\left\{ \begin{array}{l} M \to M \\ g \mapsto s \cdot g \end{array} \right\}$ is an injective G-module homomorphism, we have

$$\dim_G(A_n f) = \dim_G(s^n A_n f).$$

By Lemma 7,

$$
\begin{aligned}
s^n A_n f &= \sum_{\substack{k\in\mathbb{N},\beta\in\mathbb{N}^{d+e}\\ k+|\beta|\le n}} s^n\cdot R_k\cdot\partial^\beta f\\
&= \sum_{\substack{k,\beta\\ k+|\beta|\le n}} R_k\cdot s^{n-|\beta|}\cdot s^{|\beta|}\partial^\beta f\\
&\subset \sum_{\substack{k,\beta\\ k+|\beta|\le n}} R_k R_{r(n-|\beta|)}\cdot\sum_{\alpha\in J} R_{r|\beta|}\partial^\alpha f\\
&\subset \sum_{\alpha\in J}\left(\sum_{\substack{k,\beta\\ k+|\beta|\le n}} R_k R_{r(n-|\beta|)} R_{r|\beta|}\right)\partial^\alpha f\\
&\subset \sum_{\alpha\in J}\left(\sum_{k\le n} R_{k+rn}\right)\partial^\alpha f\\
&= \sum_{\alpha\in J} R_{(r+1)n}\partial^\alpha f
\end{aligned}
$$

Thus $\dim_G(s^n A_n f)\le \#J\cdot\dim_G R_{(r+1)n} = \#J\cdot\binom{(r+1)n+d}{d} = O\big(n^d\big)$.

"$\Leftarrow$" For any $\partial\in X$ we can apply Lemma 6 to $A' := R[\partial]$ since

$$
\dim_G(A'\cap A_n) = \dim_G \bigoplus_{|\alpha|+k\le n} G\cdot z^\alpha\partial^k = \binom{n+d+1}{d+1} \sim \frac{n^{d+1}}{(d+1)!}
$$

and $\dim_G(A'\cap A_n)f \le \dim_G A_n f = O\big(n^d\big)$ for $n\gg 0$. We thus get a differential operator $L\in R[\partial]\setminus\{0\}$ with $Lf=0$ and have verified that $f\in D(R,M,\{\partial\})$. By Prop. 5 we have $f\in D(R,M,X)$.

q.e.d.

4. Substitutions

Let M be a torsion-free X-differential R-module. By Prop. 1, $M[\![y_1,\dots,y_r]\!]$ is an $(X\cup Y)$-differential $R[y_1,\dots,y_r]$-module, and $M[\![z_1,\dots,z_s]\!]$ is an $(X\cup Z)$-differential $R[z_1,\dots,z_s]$-module. For short, we write $R[Y]$ for $R[y_1,\dots,y_r]$ etc. and put $R(Z) := \mathrm{Quot}\,(R[Z])$.
Let $N \le M[\![Y]\!]$ be an $R[Y]$-sub-differential-module. We call $\sigma = (y_1 \leftarrow \sigma_1,\dots,y_r \leftarrow \sigma_r)$ with all $\sigma_i \in R[\![Z]\!]$ a *valid substitution* on N if

$$\sigma^* : \left\{ \begin{array}{ccc} N & \to & M[\![Z]\!] \\ g(y_1,\dots,y_r) & \mapsto & g(\sigma_1,\dots,\sigma_r) \end{array} \right\}$$

is well defined. (If $N = M[Y]$, any σ is valid. If $N = M[\![Y]\!]$, σ is valid if $\sigma_i(0,\dots,0) = 0$ for all i.)

THEOREM 9:
If $\sigma_1,\dots,\sigma_r$ are algebraic over $R(Z)$ and $f \in D(R[Y], N, X\cup Y)$, then $\sigma^*(f) \in D(R[Z], M[\![Z]\!], X\cup Z)$.

Proof:
(Cf. [Hai] 2.13.) σ also induces a ring homomorphism

$$\sigma_R^* : \left\{ \begin{array}{cc} R[Y] & \to R[\sigma_1,\dots,\sigma_r] \subset R[\![Z]\!] \\ g(y_1,\dots,y_r) \mapsto & g(\sigma_1,\dots,\sigma_r) \end{array} \right\}.$$

1. Case 1: σ_R^* is injective. Then it induces a field homomorphism $\sigma_R^* : R(Y) \to K$, where

   $$K := R(Z)(\sigma_1,\dots,\sigma_r)$$

   is a finite field extension of $R(Z)$ by the assumption. Because of characteristic 0, the derivations $\partial \in X\cup Z$ can be extended from $R(Z)$ to K, so that $\partial(\sigma_i) \in K$ for all i. By the chain rule, we have for any $\partial \in X\cup Z$ and any $g \in (R[Y]\setminus\{0\})^{-1}N$:

   $$\partial(\sigma^*(g)) = \sum_{i=1}^{r} \partial(\sigma_i)\cdot\sigma^*(\partial_{y_i}(g)) + \left\{ \begin{array}{ll} \sigma^*(\partial(g)) & \text{if } \partial \in X \\ 0 & \text{if } \partial \in Z \end{array} \right\}$$

   (this equation being valid in $(R[Z]\setminus\{0\})^{-1}M[\![Z]\!]$). This shows that $\sum_{\alpha\in\mathbb{N}^{\#X+r}} K\cdot\sigma^*({\partial_{X\cup Y}}^{\alpha}(f))$ is closed with respect to taking a derivative $\partial \in X\cup Z$, giving

   $$A_{X\cup Z}(\sigma^*(f)) \subset \sum_{\alpha} K\cdot\sigma^*({\partial_{X\cup Y}}^{\alpha}(f)).$$

   $A_{X\cup Y}(f)$ is finitely generated over $R(Y)$ by the assumption: $A_{X\cup Y}(f) = \sum_{\alpha\in J} R(Y)\cdot{\partial_{X\cup Y}}^{\alpha}(f)$ with $\#J < \infty$. Applying σ^*, it follows that

   $$\sum_{\alpha} K\cdot\sigma^*({\partial_{X\cup Y}}^{\alpha}(f)) = \sum_{\alpha\in J} K\cdot\sigma^*({\partial_{X\cup Y}}^{\alpha}(f))$$

   and therefore

   $$\begin{aligned} \dim_{R(Z)} A_{X\cup Z}(\sigma^*(f)) &\le \dim_{R(Z)} \sum_{\alpha} K\cdot\sigma^*({\partial_{X\cup Y}}^{\alpha}(f)) \\ &\le \dim_{R(Z)} K\cdot\#J < \infty \end{aligned}$$

2. Case 2: $s = r-1$ and $\sigma_1 = z_1,\dots,\sigma_s = z_s, \sigma_r = 0$. Then σ^* simply puts a zero in the last coordinate. For every $\partial \in X\cup\{\partial_{y_1},\dots,\partial_{y_s}\}$ there is a nonzero differential operator $L \in R[Y][\partial]$ with $Lf = 0$. Write L as $L = \sum_{j\in\mathbb{N}} {y_r}^j L_j$ with $L_j \in R[y_1,\dots,y_s][\partial]$. Since $M[\![Y]\!]$ is a torsion-free $R[Y]$-module, we can replace L by $\frac{1}{y_r}L$ until we reach $L_0 \neq 0$. Then $L_0\sigma^*(f) = \sigma^*(L)\sigma^*(f) = \sigma^*(Lf) = 0$.
3. Case 3: $s \le r$ and $\sigma_1 = z_1,\dots,\sigma_s = z_s$, $\sigma_{s+1} = 0,\dots,\sigma_r = 0$. This is done by applying case 2 $r-s$ times.

4. Case 4: the general case. Split

$$g(y_1,\dots,y_r)\overset{\sigma^*}{\mapsto}g(\sigma_1(Z),\dots,\sigma_r(Z))$$

into

$$g(y_1,\dots,y_r)\overset{\tau^*}{\mapsto}g(\sigma_1(Z)+w_1,\dots,\sigma_r(Z)+w_r)\overset{\nu^*}{\mapsto}g(\sigma_1(Z),\dots,\sigma_r(Z))$$

where

$w_1,\dots,w_r$ are new variables,
$\tau=(y_1\leftarrow\sigma_1(Z)+w_1,\dots,y_r\leftarrow\sigma_r(Z)+w_r)$ and
$\nu=(z_1\leftarrow z_1,\dots,z_r\leftarrow z_r,w_1\leftarrow 0,\dots,w_r\leftarrow 0)$.
τ^* is injective (the inverse substitution is $(z_1\leftarrow 0,\dots,z_r\leftarrow 0,w_1\leftarrow y_1-\sigma_1(0),\dots,w_r\leftarrow y_r-\sigma_r(0))$), and the result follows by applying the cases 1 and 3.

q.e.d.

5. The Diagonal Theorem

THEOREM 10:
The diagonal theorem. Let H be a torsion-free Y-differential G-module. Put $R:=G[z_1,\dots,z_d]$, $M:=H[\![z_1,\dots,z_d]\!]$ and $X:=Y\cup Z$. By Prop. 1, M is an X-differential R-module. Given $a_1,\dots,a_d\in\mathbb{Z}$ and $b\in\mathbb{Z}$ we form the equation $a_1\alpha_1+\dots+a_d\alpha_d=b$, which we also write as $a^\top\alpha=b$, and the diagonal operator

$$\nabla:\left\{\begin{matrix} M & \to & M \\ f=\sum\limits_{\alpha\in\mathbb{N}^d}f_\alpha z^\alpha & \mapsto & \nabla f=\sum\limits_{\substack{\alpha\in\mathbb{N}^d\\ a^\top\alpha=b}}f_\alpha z^\alpha\end{matrix}\right\}$$

which cuts out a certain "diagonal" from f. The claim is: If $f\in D(R,M,X)$, then $\nabla f\in D(R,M,X)$.

The name "diagonal theorem" comes from the following

COROLLARY 11:
Let M be a torsion-free X-differential R-module and

$$f=\sum_{\alpha,\beta\in\mathbb{N}}f_{\alpha,\beta}y^\alpha z^\beta\in D(R[y,z],M[\![y,z]\!],X\cup\{\partial_y,\partial_z\}).$$

Then

$$\mathrm{diag}f:=\sum_{\alpha\in\mathbb{N}}f_{\alpha,\alpha}y^\alpha\in D(R[y],M[\![y]\!],X\cup\{\partial_y\}).$$

Proof of the Corollary:
The equation for the exponents is $\alpha-\beta=0$.
By Theorem 10, $\nabla_{\alpha-\beta}(f)=\sum_{\alpha\in\mathbb{N}}f_{\alpha,\alpha}y^\alpha z^\alpha$ is in $D(R[y,z],N,X\cup\{\partial_y,\partial_z\})$. where N denotes the sub-$(X\cup\{\partial_y,\partial_z\})$-differential $R[y,z]$-module $M[y,z][\![yz]\!]$ of $M[\![y,z]\!]$. Apply the substitution $\sigma=(z\leftarrow 1)$ which is well-defined on N. The claim then follows from Theorem 9.
Proof of the Theorem:
(Our statement of the diagonal theorem is more general than the one proved in [Li1]. The following proof grew out of the proof given in [Hai] 2.18.)
We may assume that $\mathbb{Q}\subset G$, otherwise we replace G by $(\mathbb{Z}\setminus\{0\})^{-1}G$ and H by $(\mathbb{Z}\setminus\{0\})^{-1}H$.
If either all the $a_i\geq 0$ or all the $a_i\leq 0$, we may assume all the $a_i\geq 0$. Renumbering the variables, we may assume $a_1,\dots,a_k>0$ and $a_{k+1}=\dots=a_d=0$. Then the solution set of the equation is $\left\{\alpha\in\mathbb{N}^d:a^\top\alpha=b\right\}=I\times\mathbb{N}^{n-k}$ where $I\subset\mathbb{N}^k$ is a finite set. Now

$$\nabla f=\sum_{(\alpha_1,\dots,\alpha_k)\in I}\ \sum_{\alpha_{k+1},\dots,\alpha_d\in\mathbb{N}}f_\alpha z^\alpha=\sum_{\beta\in I}(z_1\leftarrow 0,\dots,z_k\leftarrow 0)^*\left(\frac{1}{\beta!}\partial_{z_1}{}^{\beta_1}\cdots\partial_{z_k}{}^{\beta_k}f\right)$$

is D-finite by Theorem 9.
We can now assume that there is some $a_i > 0$ and some $a_i < 0$.
Let $A := R[Y,Z] = R[\partial_{z_1},\dots,\partial_{z_d},\partial_{-1},\dots,\partial_{-e}] = G[z_1,\dots,z_d,\partial_{z_1},\dots,\partial_{z_d},\partial_{-1},\dots,\partial_{-e}]$ be the WEYL algebra. By Theorem 8 we know that

$$\dim_G A_n f = O\big(n^d\big) \quad \text{for } n \gg 0 \tag{1}$$

Put $\Delta := a_1 \cdot z_1\partial_{z_1} + \dots + a_d \cdot z_d\partial_{z_d} - b \in A$. For $f = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha$ we have

$$\Delta f = \sum_{\alpha\in\mathbb{N}^d} \big(a^\top\alpha - b\big) f_\alpha z^\alpha \tag{2}$$

$$\nabla f = \sum_{\substack{\alpha\in\mathbb{N}^d\\ a^\top\alpha = b}} f_\alpha z^\alpha \tag{3}$$

Δ and ∇ are complementary in a certain sense:

$$\nabla\Delta f = 0 \tag{4}$$

$$\Delta\nabla f = 0 \tag{5}$$

$$\Delta f = 0 \;\Leftrightarrow\; f = \nabla f \tag{6}$$

By (5) and (6) we have

$$\nabla\nabla f = \nabla f \tag{7}$$

Furthermore

$$\Delta M = \big\{f \in M : f_\alpha = 0 \text{ if } a^\top\alpha = b\big\} = \ker\nabla \tag{8}$$

$$\nabla M = \big\{f \in M : f_\alpha = 0 \text{ if } a^\top\alpha \neq b\big\} = \ker\Delta^k \quad \text{for every } k \geq 1 \tag{9}$$

Our first important claim is:

$$\text{For } L \in A \text{ with } \Delta L = L\Delta \text{ and } f \in M\text{:} \qquad \nabla L f = L\nabla f \tag{10}$$

Proof: By (7) and (8), we can write $f = \nabla f + \Delta\widetilde{f}$ with $\widetilde{f} \in M$. Because of $\Delta L = L\Delta$ and (4), $\nabla L f = \nabla L\nabla f + \nabla L\Delta\widetilde{f} = \nabla L\nabla f + \nabla\Delta L\widetilde{f} = \nabla L\nabla f$. Because of $\Delta L = L\Delta$ and (5), $\Delta L\nabla f = L\Delta\nabla f = 0$, so (6) implies $\nabla L\nabla f = L\nabla f$.
Let $A^\Delta := \{L \in A : \Delta L = L\Delta\}$ be the set of those differential operators that commute with Δ. A^Δ is a subalgebra of A. We need an explicit description of A^Δ: Writing $L = \sum_{\substack{\alpha\in\mathbb{N}^d\\ \beta\in\mathbb{N}^{d+e}}} c_{\alpha,\beta} z^\alpha\partial^\beta$ with all $c_{\alpha,\beta} \in G$, we compute

$$\begin{aligned}
\Delta L - L\Delta &= \sum_{\alpha,\beta} c_{\alpha,\beta}\big(\Delta z^\alpha\partial^\beta - z^\alpha\partial^\beta\Delta\big)\\
&= \sum_{\alpha,\beta} c_{\alpha,\beta} \sum_{i=1}^d a_i\big(z_i\partial_{z_i} z^\alpha\partial^\beta - z^\alpha\partial^\beta z_i\partial_{z_i}\big)\\
&= \sum_{\alpha,\beta} c_{\alpha,\beta} \sum_{i=1}^d a_i\Big((z_i z^\alpha\partial_{z_i} + \alpha_i z^\alpha)\partial^\beta - z^\alpha\big(z_i\partial^\beta\partial_{z_i} + \beta_i\partial^\beta\big)\Big)\\
&= \sum_{\alpha,\beta} c_{\alpha,\beta} \sum_{i=1}^d a_i(\alpha_i - \beta_i) z^\alpha\partial^\beta\\
&= \sum_{\alpha,\beta} c_{\alpha,\beta} \cdot a^\top(\alpha-\beta)\cdot z^\alpha\partial^\beta
\end{aligned}$$

Therefore

$$A^\Delta = \bigoplus_{\substack{\alpha\in\mathbb{N}^d \\ (\beta,\gamma)\in\mathbb{N}^{d+e} \\ a^\top(\alpha-\beta)=0}} G\cdot z^\alpha\partial^{(\beta,\gamma)} \tag{11}$$

Fix a derivation $\partial\in X$. Our aim is to find an equation $L\nabla f=0$ with $L\in R[\partial]\setminus\{0\}$.
Let $B:=\{L\in R[\partial]:\Delta L=L\Delta\}=R[\partial]\cap A^\Delta$, a subalgebra of A. We claim

(i) $\dim_G(B\cap A_n)\geq cn^d$ for $n\gg 0$ and some $c>0$.

(ii) $B[\Delta]=\bigoplus_{k\in\mathbb{N}}\Delta^k B$.

(iii) $\dim_G(B[\Delta]\cap A_n)\geq cn^{d+1}$ for $n\gg 0$ and some $c>0$.

(iv) $\dim_G(B[\Delta]\cap A_n)f=O\big(n^d\big)$ for $n\gg 0$.

Proof:

(i) By (11) we have $B=\bigoplus_{\substack{\alpha\in\mathbb{N}^d\\ \beta\in\mathbb{N}\\ a^\top\alpha=0}} G\cdot z^\alpha\partial^\beta$ if $\partial\in Y$ and $B=\bigoplus_{\substack{\alpha\in\mathbb{N}^d\\ \beta\in\mathbb{N}\\ a^\top\alpha=a_j\beta}} G\cdot z^\alpha\partial^\beta$ if $\partial=\partial_{z_j}\in Z$. There are $(d-1)$ $\mathbb{Z}$-linear independent solutions $\alpha^{(2)},\ldots,\alpha^{(d)}\in\mathbb{N}^d$ of the equation $a^\top\alpha=0$. (To see this: We may assume $a_1>0$ and $a_2<0$, else renumber variables. Put $\alpha^{(2)}:=(-a_2,a_1,0\ldots 0)$, and for $i\in\{3,\ldots,d\}$: $\alpha^{(i)}:=\left\{\begin{array}{ll}(0,0,0\ldots 0,1,0\ldots 0) & \text{if } a_i=0\\ (-a_i,0,0\ldots 0,a_1,0\ldots 0) & \text{if } a_i<0\\ (0,a_i,0\ldots 0,-a_2,0\ldots 0) & \text{if } a_i>0\end{array}\right\}$, with non-zero entries only in the columns 1,2,i.) Now put $\beta^{(2)}=\cdots=\beta^{(d)}:=0$, $\beta^{(1)}:=1$ and $\alpha^{(1)}:=\left\{\begin{array}{ll}(0\ldots 0) & \text{if } \partial\in Y\\ (0\ldots 0,1,0\ldots 0) & \text{if } \partial=\partial_{z_j}\in Z\end{array}\right\}$. We thus have d $\mathbb{Z}$-linear independent solutions $\big(\alpha^{(1)},\beta^{(1)}\big),\ldots,\big(\alpha^{(d)},\beta^{(d)}\big)\in\mathbb{N}^{d+1}$ of the equation

$$\left\{\begin{array}{ll}a^\top\alpha=0 & \text{if } \partial\in Y\\ a^\top\alpha=a_j\beta & \text{if } \partial=\partial_{z_j}\in Z\end{array}\right\} \tag{12}$$

The map

$$\varphi:\left\{\begin{array}{ccc}\mathbb{N}^d & \to & \mathbb{N}^{d+1}\\ (\gamma_1,\ldots,\gamma_d) & \mapsto & \sum_{i=1}^d\gamma_i\cdot\big(\alpha^{(i)},\beta^{(i)}\big)\end{array}\right\}$$

is injective, and for $|\gamma|\leq m$ we have, putting $h:=\max\big\{\big|\alpha^{(i)}\big|+\beta^{(i)}:i\in\{1,\ldots,d\}\big\}>0$:

$$|\varphi(\gamma)|=\left|\left(\sum_i\gamma_i\alpha^{(i)},\sum_i\gamma_i\beta^{(i)}\right)\right|=\sum_i\gamma_i\Big(\big|\alpha^{(i)}\big|+\beta^{(i)}\Big)\leq\sum_i\gamma_i\cdot h\leq m\cdot h$$

Thus $\varphi\Big(\Big\{\gamma\in\mathbb{N}^d:|\gamma|\leq m\Big\}\Big)\subset\Big\{(\alpha,\beta)\in\mathbb{N}^{d+1}:(12)\text{ holds},|\alpha|+\beta\leq mh\Big\}$.
For $mh\leq n<(m+1)h$ we therefore have

$$\begin{aligned}\dim_G(B\cap A_n)&=\#\Big\{(\alpha,\beta)\in\mathbb{N}^{d+1}:(12)\text{ holds},|\alpha|+\beta\leq mh\Big\}\\ &\geq\#\varphi\Big(\Big\{\gamma\in\mathbb{N}^d:|\gamma|\leq m\Big\}\Big)\\ &=\binom{m+d}{d}=\binom{\lfloor\frac{n}{h}\rfloor+d}{d}\\ &\geq c\cdot n^d\quad\text{for } n\gg 0\end{aligned}$$

(ii) There is an $i\in\{1,\ldots,d\}$ such that $\partial\neq\partial_{z_i}$ and $a_i\neq 0$. Let $\deg: A\to\mathbb{N}\cup\{-\infty\}$ be the degree in ∂_{z_i} (well-defined on A!). Then $\deg\Delta=1$, $\deg\Delta^kL=k+\deg L$ for $L\in A$, and $\deg\Delta^0L_0+\cdots\Delta^kL_k=k$ for any $L_0,\ldots,L_k\in B$ with $L_k\neq 0$.

(iii) By (ii) and since the multiplication by Δ^k is an injective G-linear map $A_{n-k} \to A_n$

$$\begin{aligned}
\dim_G (B[\Delta] \cap A_n) &\geq \dim_G \left(\bigoplus_{k \in \mathbb{N}} (\Delta^k B \cap A_n) \right) \\
&= \sum_{k \in \mathbb{N}} \dim_G \left(\Delta^k B \cap A_n \right) \\
&\geq \sum_{k=0}^{n} \dim_G (B \cap A_{n-k}) \\
&\geq \sum_{k=0}^{n-n_0} c(n-k)^d = \sum_{k=n_0}^{n} ck^d \\
&\sim \frac{c}{d+1} n^{d+1}
\end{aligned}$$

(iv) Follows from (1).
Let $\pi : B[\Delta] \to B$ be the G-algebra homomorphism defined by $\pi|_B = \mathrm{id}_B$ and $\pi(\Delta) = 0$. (That is, $\pi\left(\sum_{k \in \mathbb{N}} \Delta^k L_k\right) = L_0$.) We claim that

$$\text{for } L \in B[\Delta] \text{ and } f \in M: \quad \nabla L f = \pi(L) \nabla f \tag{13}$$

Proof: Write $L = \pi(L) + \Delta \widetilde{L}$ with $\widetilde{L} \in B[\Delta]$. By (4), $\nabla L f = \nabla \pi(L) f + \nabla \Delta \widetilde{L} f = \nabla \pi(L) f$. Because of $\pi(L) \in A^\Delta$ and (10), $\nabla \pi(L) f = \pi(L) \nabla f$.
By (iii), (iv) and Lemma 6 we have a differential operator $L \in B[\Delta] \setminus \{0\}$ with $Lf = 0$. By (13), $\pi(L) \nabla f = 0$. If $\pi(L) \neq 0$, we are done since $B \subset R[\partial]$. Else write L uniquely as $L = \Delta^k L'$ with $k \geq 1$, $L' \in B[\Delta]$ and $\pi(L') \neq 0$. By (9) and (6), $L' f \in \ker \Delta^k = \ker \Delta$, so $\nabla L' f = L' f$. By Prop. 3(g), $f \in D(R, M, X)$ implies $L' f \in D(R, M, X)$, so there is an $L'' \in R[\partial] \setminus \{0\}$ with $L'' L' f = 0$. Then $L'' \pi(L') \nabla f = L'' \nabla L' f = L'' L' f = 0$ by (13), and we are done because $L'' \pi(L') \in R[\partial] \setminus \{0\}$.
We have shown that for every $\partial \in X$ there is an $L \in R[\partial] \setminus \{0\}$ with $L \nabla f = 0$. Now ∇f is D-finite by Theorem 8. **q.e.d.**

6. Combinatorial Sums

Let R be a field of characteristic zero.

Definition 7:

An expression (depending on $n_1, \ldots, n_e \in \mathbb{N}$) of the form

$$S(n_1, \ldots, n_e) = \sum_{\substack{n_{e+1}, \ldots, n_d \in \mathbb{N} \\ An \overset{\geq}{\underset{<}{=}} b}} P_1(n_1) \cdots P_d(n_d) \cdot \prod_{k=1}^{n_1} Q_1(k) \cdots \prod_{k=1}^{n_d} Q_d(k)$$

(n denotes the column vector $\begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix}$) with

- $A \in \mathbb{Z}^{c \times d}$, $b \in \mathbb{Z}^d$, and $\overset{\geq}{\underset{<}{=}}$ a fixed set of c equality/inequality signs $\in \{=, \leq, \geq, <, >\}$,
- $P_1, \ldots, P_d \in R(X)$ rational functions without poles in $\mathbb{N}$,
- $Q_1, \ldots, Q_d \in R(X)$ rational functions without poles in $\mathbb{N} \setminus \{0\}$,
- such that for every $(n_1, \ldots, n_e) \in \mathbb{N}^e$ there are only finitely many solutions $(n_{e+1}, \ldots, n_d) \in \mathbb{N}^{d-e}$ of $An \overset{\geq}{\underset{<}{=}} b$

is called a *combinatorial sum* over R.

Theorem 12:

The generating function

$$\sum_{n_1, \ldots, n_e \in \mathbb{N}} S(n_1, \ldots, n_e) {z_1}^{n_1} \cdots {z_e}^{n_e}$$

of a combinatorial sum is in $D(R[z_1, \ldots, z_e], R[\![z_1, \ldots, z_e]\!], \{\partial_{z_1}, \ldots, \partial_{z_e}\})$.

Corollary 13:

Any combinatorial sum of one variable $S(m)$ over R satisfies a non-trivial linear recurrence relation with polynomial coefficients: There are polynomials $a_r, \ldots, a_0 \in R[m]$, not all zero, such that

$$a_r(m) \cdot S(m + r) + \cdots + a_0(m) \cdot S(m) = 0 \quad \forall m \in \mathbb{N}$$

For the proof, we recall the classical

Proposition 14:

Let R be an integral domain of characteristic zero, M an R-module, and $f = \sum\limits_{m=0}^{\infty} a_m z^m \in M[\![z]\!]$.

Then $f \in D(R[z], M[\![z]\!], \{\partial_z\})$ if and only if the sequence (a_m) satisfies a non-trivial linear recurrence relation with polynomial coefficients.

Proof of the Proposition:

See [Hai], section 1.14 or [Sta], Theorem 1.5.

Proof of the Theorem:

We may assume that we only have equality signs. (Introduce an auxiliary variable for every inequality.) The unrestricted sum

$$\sum_{n_1, \ldots, n_d \in \mathbb{N}} \prod_{j=1}^{d} \left(P_j(n_j) \cdot \prod_{k=1}^{n_j} Q_j(k) \right) \cdot z^n = \prod_{j=1}^{d} \left(\sum_{m=0}^{\infty} P_j(m) \cdot \prod_{k=1}^{m} Q_j(k) \cdot {z_j}^m \right)$$

is in $D(R[z_1, \ldots, z_d], R[\![z_1, \ldots, z_d]\!], \{\partial_{z_1}, \ldots, \partial_{z_d}\})$ by Prop. 14 and Prop. 5(a). Applying Theorem 10 c times, we get

$$\sum_{\substack{n_1, \ldots, n_d \in \mathbb{N} \\ An = b}} \prod_{j=1}^{d} \left(P_j(n_j) \cdot \prod_{k=1}^{n_j} Q_j(k) \right) \cdot z^n \in D(R[z_1, \ldots, z_d], R[\![z_1, \ldots, z_d]\!], \{\partial_{z_1}, \ldots, \partial_{z_d}\})$$

Then apply the substitution $(z_{e+1} \leftarrow 1, \ldots, z_d \leftarrow 1)$ which is well-defined on the sub-$\{\partial_{z_1}, \ldots, \partial_{z_d}\}$-differential $R[z_1, \ldots, z_d]$-module

$$N := \left\{ \sum_{n \in \mathbb{N}^d} a_n z^n : \text{there is an } l \in \mathbb{N} \text{ such that } a_n = 0 \text{ if } \|An - b\| > l \right\}$$

of $R[\![z_1, \ldots, z_d]\!]$. Theorem 9 now gives the desired result. **q.e.d.**

Proof of the Corollary:

Put $e = 1$ in Theorem 12 and use Prop. 14. **q.e.d.**

7. Notes

Section 1:
The idea for the axiomatic approach comes from [Ko].
Section 2:
Definition 4 was first formulated by [Sta] (for $d = 1$) and [Li1]. Prop. 3 and Prop. 4(a) are found in [Hai]; see also [Li2]. Prop. 4(b) comes from [Sto] and [HaSi]. Prop. 5 is mentioned in [Li1] and proved in [Li2]. [Ca] presents some more definitions of D-finiteness, based on Bernstein theory.
Section 3:
The importance of estimating the degrees of polynomials has first been recognized by Gessel [Ge], Zeilberger [Ze] and Lipshitz [Li1]. More on filtrations can be found in the books of J.-E. Björk [Bj] and A. Borel [Bo]. Theorem 8 is a generalization of previously known statements (see [Ca] or [Sto]): they handled only $e = 0$.
Section 4:
The determination of what substitutions are valid in the case $N = M[\![Y]\!]$ goes back to Gessel [Ge]. Theorem 9 is a generalization of [Hai] 2.13. (The proof in [Li2] is incomplete: it only treats case 1.)
Section 5:
Theorem 10 was first proved by Lipshitz [Li1] (case $e = 0$ only) and independently by Haible [Hai]. Both proofs involved much computation. A new proof was found by Stoll [Sto] (case $e = 0$ only). The present proof is a generalization of his proof to arbitrary e.
Section 6: A similar but less explicit statement can be found in [Li1]. P-recursive sequences (i.e. sequences whose generating function is D-finite) are treated in more detail in [Li2].

8. References

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