Asymptotic Expansions of P-recursive Sequences

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1. Abstract

Many P-recursive sequences admit an asymptotic expansion of the form

$$f_n \sim K(n) n^\theta \Big(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots\Big),$$

where $K(n)$ is a simple P-recursive leading term. In examples arising in combinatorics, one frequently observes striking integrality properties of the a_k: they are rational numbers whose denominators are made up of a finite set of primes only. In this paper we prove this for several examples, by means of an auxiliary P-recursive sequence.

2. Introduction

Recall that a sequence (f_n) of complex numbers is called *P-recursive* if

$$P_r(n)f_{n+r} + \cdots + P_0(n)f_n = 0 \quad \text{for all } n \geq 0,$$

where $P_r, \ldots, P_0 \in \mathbb{C}[n]$ are polynomials, not all zero. It is well-known (see e.g. [Sta], Theorem 1.5, or [Hai] 1.14) that this is equivalent to the fact that the generating function $f(X) = \sum_{n=0}^{\infty} f_n X^n$ is *D-finite*, i.e. satisfies a nontrivial linear differential equation with coefficients in $\mathbb{C}[X]$.

A classical result of Birkhoff and Trjitzinsky (see [Bi], [BiTr], [WiZe] and [Hai] 1.16) describes the asymptotics of the solutions of a recurrence:

Proposition 1:

Assume the recurrence relation in the form

$$P_r(n)f_{n+r} + \cdots + P_0(n)f_n = 0$$

with $P_r, \ldots, P_0 \in \mathbb{C}[n]$, $P_0 \neq 0$, and such that P_r has no zero in $\mathbb{N}$. Then there is a formal fundamental system of r formal asymptotic solutions of the form

$$\begin{aligned} a_n \sim \exp\left(\mu_0 \cdot n\log n + \mu_1 \cdot n + \mu_2 \cdot n^{\frac{N-1}{N}} + \cdots + \mu_N \cdot n^{\frac{1}{N}}\right) \\ \cdot n^{\theta} \cdot \sum_{k=0}^{t} n^{r_k/N} \cdot \left(b_{0k} n^{-\frac{0}{N}} + b_{1k} n^{-\frac{1}{N}} + \ldots\right) \cdot (\log n)^k \end{aligned} \tag{1}$$

with $N \in \mathbb{N} \setminus \{0\}$, $\mu_0 \in \frac{1}{N}\mathbb{Z}$, $\mu_1, \ldots, \mu_N \in \mathbb{C}$, $-\pi \leq \operatorname{Im} \mu_1 < \pi$, $\theta \in \mathbb{C}$, $t \in \mathbb{N}$, all $b_{jk} \in \mathbb{C}$, and $b_{0k} = 0$ only if $b_{0k} = b_{1k} = \cdots = 0$. To each of these r formal asymptotic solutions corresponds an actual solution of the recurrence which has the desired asymptotics. These r solutions form a fundamental system (basis of the solution space) of the recurrence.

3. The main transformation

Let $N \in \mathbb{N} \setminus \{0\}$ be fixed. We consider generating functions

$$f(X) = \sum_{n \gg -\infty} f_n X^n$$

such that there is a sequence $(a_k)_{k \gg -\infty}$ in $\mathbb{C}$ with the property that for all $m \in \mathbb{Z}$:

$$f_n = \sum_{k<m} a_k n^{-\frac{k}{N}} + O\big(n^{-\frac{m}{N}}\big) \quad \text{as } n \to \infty.$$

We then say that the sequence (f_n) has the *asymptotic expansion*

$$f_n \sim \sum_k a_k n^{-\frac{k}{N}}.$$

Let $\mathcal{F}$ be the vector space of such functions f, and let $\mathcal{A} = \mathbb{C}((T^{\frac{1}{N}}))$.

THEOREM 2:

(a) The operator

$$\Phi : \left\{ \begin{array}{ll} \mathcal{F} \to & \mathcal{A} \\ f \mapsto & \sum\limits_{k \in \mathbb{Z}} a_k \frac{1}{\Gamma(\frac{k}{N})} T^{\frac{k}{N}-1} \end{array} \right\},$$

a linear map which maps every function to its (appropriately encoded) asymptotic development, has the following properties:

(i) $f(X)$ a polynomial $\Rightarrow$ $\Phi(f) = 0$.

(ii) $\Phi(Xf') = \frac{d}{dT}\Phi(f)$.

(iii) $\Phi(Xf) = e^T \cdot \Phi(f)$.

(b) Let σ be the substitution $\sigma : \left\{ \begin{array}{ll} \mathbb{C}((T^{\frac{1}{N}})) \to & \mathbb{C}((X^{\frac{1}{N}})) \\ T & \mapsto \log(1+X) \end{array} \right\}$ and $\Psi := \sigma \circ \Phi$. Then Ψ has the following properties:

(i) $f(X)$ a polynomial $\Rightarrow$ $\Psi(f) = 0$.

(ii) $\Psi(Xf) = (X+1) \cdot \Psi(f)$.

(iii) $\Psi(f') = \Psi(f)'$.

Proof:

(a) Note that $\Phi(f)$ is well-defined because $\frac{1}{\Gamma(\alpha)}$ is always finite.

(i) If $f_n = 0$ for $n \gg 0$ then all $a_k = 0$.

(ii) Writing $f(X) = \sum_n f_n X^n$, we have $Xf'(X) = \sum_n n f_n X^n$. From

$$f_n = \sum_{k<m+N} a_k n^{-\frac{k}{N}} + O\left(n^{-\frac{m+N}{N}}\right) \quad \text{for } n \to \infty \quad \forall m \in \mathbb{Z}$$

Notation: $n \gg -\infty$ means $n \geq n_0$ for some $n_0 > -\infty$.

we get

$$\begin{aligned} nf_n &= \sum_{k<m+N} a_k n^{-\frac{k}{N}+1} + O\big(n^{-\frac{m}{N}}\big) \\ &= \sum_{k<m} a_{k+N} n^{-\frac{k}{N}} + O\big(n^{-\frac{m}{N}}\big) \quad \text{for } n \to \infty \quad \forall m \in \mathbb{Z}. \end{aligned}$$

Therefore

$$\begin{aligned} \Phi(Xf'(X)) &= \sum_k a_{k+N} \frac{1}{\Gamma\big(\frac{k}{N}\big)} T^{\frac{k}{N}-1} = \sum_k a_k \frac{1}{\Gamma\big(\frac{k}{N}-1\big)} T^{\frac{k}{N}-2} \\ &= \sum_k a_k \frac{1}{\Gamma\big(\frac{k}{N}\big)} \left(\frac{k}{N}-1\right) T^{\frac{k}{N}-2} = \frac{d}{dT}\Phi(f). \end{aligned}$$

(iii) Writing $f(X) = \sum_n f_n X^n$, we have $Xf(X) = \sum_n f_{n-1} X^n$, therefore

$$\begin{aligned} f_{n-1} &= \sum_{j<m} a_j (n-1)^{-\frac{j}{N}} + O\Big((n-1)^{-\frac{m}{N}}\Big) \\ &= \sum_{j<m} a_j n^{-\frac{j}{N}} \left(1 - \frac{1}{n}\right)^{-\frac{j}{N}} + O\big(n^{-\frac{m}{N}}\big) \\ &= \sum_{j<m} a_j n^{-\frac{j}{N}} \left(\sum_{0 \le i < \frac{m-j}{N}} \frac{\frac{j}{N} \cdots \big(\frac{j}{N}+i-1\big)}{i!} \frac{1}{n^i} + O\Big(n^{-\frac{m-j}{N}}\Big) \right) + O\big(n^{-\frac{m}{N}}\big) \\ &= \sum_{\substack{j<m \\ 0 \le i < \frac{m-j}{N}}} a_j \frac{\frac{j}{N} \cdots \big(\frac{j}{N}+i-1\big)}{i!} n^{-\frac{j}{N}-i} + O\big(n^{-\frac{m}{N}}\big) \quad \forall m \in \mathbb{Z}. \end{aligned}$$

This implies

$$\begin{aligned} \Phi(Xf(X)) &= \sum_k \left(\sum_{j+Ni=k} a_j \frac{\frac{j}{N} \cdots \big(\frac{j}{N}+i-1\big)}{i!} \right) \frac{1}{\Gamma\big(\frac{k}{N}\big)} T^{\frac{k}{N}-1} \\ &= \sum_{\substack{i \in \mathbb{N} \\ j \in \mathbb{Z}}} a_j \frac{1}{\Gamma\big(\frac{j}{N}\big)} \frac{1}{i!} T^{\frac{j}{N}+i-1} \\ &= e^T \cdot \Phi(f). \end{aligned}$$

(b)

(i) Follows from (a)(i).
(ii) Follows from (a)(iii).
(iii) Follows from (a)(ii), using the chain rule:

$$\begin{aligned} \frac{d}{dX}\Psi(f) &= \frac{d}{dX}\sigma(\Phi(f)) = \frac{1}{1+X}\sigma\left(\frac{d}{dT}\Phi(f)\right) = \frac{1}{1+X}\sigma(\Phi(Xf')) \\ &= \frac{1}{1+X}\Psi(Xf') = \Psi(f'). \end{aligned}$$

q.e.d.

THEOREM 3:

(a) If $f \in \mathcal{F}$ is D-finite (i.e. $f \in D\Big(\mathbb{C}[X], \mathbb{C}((X^{\frac{1}{N}}))\Big)$), then $a := \Phi(f)$ is D-finite over e^T (i.e. $a \in D\Big(\mathbb{C}\big[e^T\big], \mathbb{C}((T^{\frac{1}{N}}))\Big)$). In fact, if

$$P_s(X) \cdot \left(X \frac{d}{dX}\right)^s f(X) + \cdots + P_0(X) \cdot f(X) \in \mathbb{C}[X]$$

then

$$P_s\big(e^T\big) \cdot \left(\frac{d}{dT}\right)^s a(T) + \cdots + P_0\big(e^T\big) \cdot a(T) = 0.$$

(b) If $f \in \mathcal{F}$ is D-finite (i.e. $f \in D\Big(\mathbb{C}[X], \mathbb{C}((X^{\frac{1}{N}}))\Big)$), then $a := \Phi(f)$ has the form $a(T) = b\big(e^T - 1\big)$ where b is D-finite (i.e. $b \in D\Big(\mathbb{C}[X], \mathbb{C}((X^{\frac{1}{N}}))\Big)$). In fact, if

$$Q_s(X) \cdot \left(\frac{d}{dX}\right)^s f(X) + \cdots + Q_0(X) \cdot f(X) \in \mathbb{C}[X]$$

then

$$Q_s(X+1) \cdot b^{(s)}(X) + \cdots + Q_0(X+1) \cdot b(X) = 0.$$

Proof:

(a) Apply Φ and use Theorem 2(a).
(b) Apply Ψ and use Theorem 2(b).

q.e.d.

4. Generalized Stirling numbers

We want to transport integrality properties from $b(X)$ to $a(T) = b\big(e^T - 1\big)$.
If $N = 1$ and $b(X) = \sum_{n=0}^{\infty} \frac{b_n}{n!} X^n$, we use the formula ([HalHei] (4.5))

$$\frac{\big(e^T - 1\big)^n}{n!} = \sum_{k=0}^{\infty} S(k,n) \frac{T^k}{k!} \tag{2}$$

where $S(k,n) \in \mathbb{N}$ are the *Stirling numbers of the second kind*, and get $a(T) = \sum_{k=0}^{\infty} \frac{a_{k+1}}{k!} T^k$ with

$$a_{k+1} = \sum_{n=0}^{k} S(k,n) b_n.$$

Let us generalize this.

Lemma 4:
Let $N \in \mathbb{N} \setminus \{0\}$, $M \in \mathbb{Z}$, $\alpha = \frac{M}{N} \in \mathbb{Q} \setminus \{-1,-2,-3,\ldots\}$. Then

$$\big(e^T - 1\big)^\alpha = \sum_{n=0}^{\infty} s_n(\alpha) \cdot \frac{1}{(\alpha+1)\cdots(\alpha+n)} \cdot T^{\alpha+n} \tag{3}$$

with $s_n(\alpha) \in \mathbb{Q}$ and for all primes p:

$$\operatorname{ord}_p\left(s_n(\alpha)\right) \geq \left\{ \begin{array}{ll} 0 & \text{if } p \nmid N \\ -2n \cdot \operatorname{ord}_p(N) - \operatorname{ord}_p((2n)!) & \text{if } p \mid N \end{array} \right\}.$$

We emphasize that $s_n(\alpha) \in \mathbb{Z}_p$ if $p \nmid N$.

Proof:
We may assume that $\frac{M}{N}$ is in lowest terms.

$$\frac{\big(e^T - 1\big)^\alpha}{T^\alpha} = \left(1 + \left(\frac{e^T - 1}{T} - 1\right)\right)^\alpha = \sum_{m=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} \left(\frac{e^T - 1}{T} - 1\right)^m$$

is in $\mathbb{Q}[\![T]\!]$. We can therefore define $s_n(\alpha)$ by equation (3). Noticing that the terms with

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Notation: ord_p denotes the p-adic exponential valuation, $\operatorname{ord}_p(x) = \sup\{n \in \mathbb{Z} : x \in p^n \mathbb{Z}_p\}$ for $x \in \mathbb{Q}_p$.

$m > n$ do not contribute to the coefficient of T^n, we get

$$\begin{aligned}
&\frac{s_n(\alpha)}{(\alpha+1)\cdots(\alpha+n)} = \\
&= \sum_{m=0}^{n} \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} \cdot [T^n] \left(\frac{e^T-1}{T} - 1\right)^m \\
&= \sum_{m=0}^{n} \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \cdot [T^n] \left(\frac{e^T-1}{T}\right)^k \\
&= \sum_{0\le k\le m\le n} \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} \binom{m}{k} (-1)^{m-k} S(n+k,k) \frac{k!}{(n+k)!},
\end{aligned}$$

that is

$$s_n(\alpha) = \sum_{0\le k\le m\le n} \frac{(\alpha+n)\cdots(\alpha-m+1)}{(n+m)!} \binom{n+m}{n+k} (-1)^{m-k} S(n+k,k).$$

If $p \nmid N$, then $\alpha \in \mathbb{Z}_p$, and we have $\mathrm{ord}_p\left((\alpha+n)\cdots(\alpha-m+1)\right) \ge \mathrm{ord}_p\left((n+m)!\right)$ (this holds for $\alpha \in \mathbb{Z}$, and throughout $\mathbb{Z}_p$ by p-adic continuity). Therefore $s_n(\alpha) \in \mathbb{Z}_p$.
If $p \mid N$, let $e = \mathrm{ord}_p(N)$. Then $\mathrm{ord}_p(\alpha+j) = -e$ for all $j \in \mathbb{Z}$. Using $m \le n$, we get

$$\mathrm{ord}_p\left(\frac{(\alpha+n)\cdots(\alpha-m+1)}{(n+m)!}\right) = (m+n)\cdot(-e) - \mathrm{ord}_p\left((n+m)!\right) \ge -2n\cdot e - \mathrm{ord}_p\left((2n)!\right)$$

and finally $\mathrm{ord}_p\left(s_n(\alpha)\right) \ge -2n\cdot e - \mathrm{ord}_p\left((2n)!\right)$. **q.e.d.**

COROLLARY 5:
Let $a(T)$ and $b(X)$ be as in Theorem 3(b). Assume

$$\Gamma\left(\frac{M}{N}+1\right)\cdot b(X) = X^{\frac{M}{N}} \cdot \sum_{n=0}^{\infty} b_n X^n$$

with $b_n \in \mathbb{Q}$. Then the coefficients $\tilde{a}_m = a_{Nm+N+M}$ of

$$\Gamma\left(\frac{M}{N}+1\right)\cdot a(T) = \sum_{m=0}^{\infty} \tilde{a}_m \frac{1}{\left(\frac{M}{N}+1\right)\cdots\left(\frac{M}{N}+m\right)} T^{\frac{M}{N}+m}$$

are $\in \mathbb{Q}$ and satisfy (p being a prime):

(a) If $p \nmid N$, then

$$\mathrm{ord}_p\left(\tilde{a}_m\right) \ge \min\left\{\mathrm{ord}_p\left(n! b_n\right) : 0 \le n \le m\right\}.$$

If $p \mid N$ and $e := \mathrm{ord}_p(N)$, then

$$\mathrm{ord}_p\left(\tilde{a}_m\right) \ge \min\left\{\mathrm{ord}_p\left(b_n\right) - (2m-n)e - \mathrm{ord}_p\left((2(m-n))!\right) : 0 \le n \le m\right\}.$$

(b) If $b_n \in \frac{1}{n!}\mathbb{Z}_p$ for all n and $p \nmid N$, then $\tilde{a}_m \in \mathbb{Z}_p$ for all m.

Proof:
From $b(X) \in X^{\frac{M}{N}} \cdot \mathbb{C}[\![X]\!]$ one gets $a(T) \in T^{\frac{M}{N}} \cdot \mathbb{C}[\![T]\!]$, therefore – in the notation of Theorem 2(a) –

$$a(T) = \sum_{m=0}^{\infty} a_{Nm+N+M} \frac{1}{\Gamma\left(\frac{M}{N} + m + 1\right)} T^{\frac{M}{N}+m}.$$

We write $\tilde{a}_m = a_{Nm+N+M}$ and $\alpha = \frac{M}{N}$. Substituting Lemma 4 into $\Gamma(\alpha+1)b(X) = \sum_{n=0}^{\infty} b_n X^{\alpha+n}$ gives

$$\Gamma(\alpha+1)a(T) = \sum_{n=0}^{\infty} b_n \sum_{k=0}^{\infty} s_k(\alpha+n) \frac{1}{(\alpha+n+1)\cdots(\alpha+n+k)} T^{\alpha+n+k}.$$

Taking the coefficient of $T^{\alpha+m}$ here, we get

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} \tilde{a}_m = \sum_{n+k=m} b_n s_k(\alpha+n) \frac{1}{(\alpha+n+1)\cdots(\alpha+n+k)}$$

or equivalently

$$\tilde{a}_m = \sum_{n=0}^{m} b_n \cdot s_{m-n}(\alpha+n) \cdot (\alpha+1)\cdots(\alpha+n).$$

(a) If $p \nmid N$ then by Lemma 4 $\mathrm{ord}_p\left(s_{m-n}(\alpha+n)\right) \geq 0$,
$\mathrm{ord}_p\left((\alpha+1)\cdots(\alpha+n)\right) \geq \mathrm{ord}_p(n!)$,
and we get $\mathrm{ord}_p(\tilde{a}_m) \geq \min\{\mathrm{ord}_p(n!b_n) : 0 \leq n \leq m\}$. If $p \mid N$ and $e = \mathrm{ord}_p(N)$, then by Lemma 4 again $\mathrm{ord}_p\left(s_{m-n}(\alpha+n)\right) \geq -2(m-n)e - \mathrm{ord}_p\left((2(m-n))!\right)$, $\mathrm{ord}_p\left((\alpha+1)\cdots(\alpha+n)\right) = -ne$, and we get
$\mathrm{ord}_p(\tilde{a}_m) \geq \min\{\mathrm{ord}_p(b_n) - (2m-n)e - \mathrm{ord}_p\left((2(m-n))!\right) : 0 \leq n \leq m\}$.

(b) Use (a) and $\mathrm{ord}_p(n!b_n) \geq 0$ for all n.

q.e.d.

5. Examples

5.1. A multinomial coefficient

PROPOSITION 6:
Let $r \in \mathbb{N} \setminus \{0\}$ and $K_r(n) = \frac{(rn)!}{n!^r}$. Then $K_r(n)$ has the asymptotic expansion

$$K_r(n) \sim \sqrt{r(2\pi)^{1-r}} \cdot (r^r)^n \cdot \sum_{k=0}^{\infty} \tilde{a}_k n^{-k}$$

with $\tilde{a}_0 = 1$ and $\tilde{a}_k \in \mathbb{Z}\left[\frac{1}{r}\right]$ for all k.

Proof:
From Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot \exp\left(\sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)} \frac{1}{n^k}\right)$$

we get

$$K_r(n) \sim r^{rn+\frac{1}{2}} \cdot (2\pi n)^{\frac{1-r}{2}} \cdot \exp\left(\sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)} \frac{1-r^{k+1}}{r^k} \frac{1}{n^k}\right)$$

that is

$$\sum_{k=0}^{\infty} \tilde{a}_k T^k = \exp\left(\sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)} \frac{1-r^{k+1}}{r^k} T^k\right).$$

We have the following

LEMMA 7:
If p is a prime and $\displaystyle\sum_{k=0}^{\infty} \frac{b_k}{k!} T^k \in \mathbb{Z}_p\left[\left[e^T - 1\right]\right]$, then $\displaystyle\exp\left(\sum_{k=1}^{\infty} \frac{b_k}{k} T^k\right) \in \mathbb{Z}_p[\![T]\!]$.

Proof:
(This is related to *Beukers' lemma*.) The maps

$$\begin{array}{ccccccc} \mathbb{Z}_p[\![X]\!] & \to & \mathbb{Q}_p[\![T]\!] & \to & \mathbb{Q}_p[\![T]\!] & \to & \mathbb{Q}_p[\![T]\!] \\ F(X) & \mapsto & F\left(e^T - 1\right) = \displaystyle\sum_{k=0}^{\infty} \frac{b_k}{k!} T^k & \mapsto & \displaystyle\sum_{k=1}^{\infty} \frac{b_k}{k} T^k & \mapsto & \exp\left(\displaystyle\sum_{k=1}^{\infty} \frac{b_k}{k} T^k\right) \end{array}$$

are continuous. We have to prove that the image of the composition map lies in $\mathbb{Z}_p[\![T]\!]$. Since $\mathbb{Z}_p[\![T]\!]$ is closed and $\mathbb{Z}_p[X]$ is dense in $\mathbb{Z}_p[\![X]\!]$, we have only to prove it for $F(X) \in$

$\mathbb{Z}_p[X] = \mathbb{Z}_p[1+X]$. Write $F(X) = \sum_{i=0}^{m} c_i (1+X)^i$ with $c_i \in \mathbb{Z}_p$. Then $\sum_{k=0}^{\infty} \frac{b_k}{k!} T^k = \sum_{i=0}^{m} c_i e^{iT}$, so $b_k = \sum_{i=0}^{m} c_i i^k$, and

$$\exp\left(\sum_{k=1}^{\infty} \frac{b_k}{k} T^k\right) = \prod_{i=0}^{m} \exp\left(\sum_{k=1}^{\infty} \frac{c_i i^k}{k} T^k\right) = \prod_{i=0}^{m} \exp\left(-c_i \log\left(1-iT\right)\right) = \prod_{i=0}^{m} \left(1-iT\right)^{-c_i}$$
$$= \prod_{i=0}^{m} \left(\sum_{k=0}^{\infty} \binom{c_i+k-1}{k} i^k T^k\right).$$

This is in $\mathbb{Z}_p[\![T]\!]$ since $\binom{c_i+k-1}{k} \in \mathbb{Z}_p$ for every $k \in \mathbb{N}$. **q.e.d.**

Proof:(of proposition 6, continued):
Let p be a prime not dividing r. Put $c := \frac{1}{r} \in {\mathbb{Z}_p}^{\times}$. The lemma is applicable to the sequence $b_k = \frac{B_{k+1}}{k+1} \frac{1-r^{k+1}}{r^k}$ since

$$\sum_{k=0}^{\infty} \frac{b_k}{k!} T^k = \sum_{k=0}^{\infty} \frac{B_{k+1}}{(k+1)!} \left(\frac{T}{r}\right)^k - r \sum_{k=0}^{\infty} \frac{B_{k+1}}{(k+1)!} T^k$$
$$= \left(\frac{1}{e^{cT}-1} - \frac{1}{cT}\right) - \frac{1}{c}\left(\frac{1}{e^T-1} - \frac{1}{T}\right) = \frac{1}{e^{cT}-1} - \frac{1}{c}\frac{1}{e^T-1} = F\left(e^T-1\right)$$

with $F(X) = \dfrac{1}{(1+X)^c - 1} - \dfrac{1}{c}\dfrac{1}{X} = \dfrac{1}{cX}\left(\dfrac{1}{1 + r\binom{c}{2}X + r\binom{c}{3}X^2 + \ldots} - 1\right) \in \mathbb{Z}_p[\![X]\!]$. The lemma now gives $\sum_{k=0}^{\infty} \tilde{a}_k T^k \in \mathbb{Z}_p[\![T]\!]$. **q.e.d.**

5.2. Refinement for $r = 2$

Proposition 8:
$K_2(n) = \frac{(2n)!}{n!^2}$ has the asymptotic expansion

$$K_2(n) \sim \frac{4^n}{\sqrt{\pi n}} \cdot \sum_{k=0}^{\infty} \tilde{a}_k n^{-k}$$

with $\tilde{a}_k \in \mathbb{Z}\left[\frac{1}{2}\right]$ and $\mathrm{ord}_2\left(\tilde{a}_k\right) \geq -3k - \mathrm{ord}_2\left(k!\right) \geq -4k$.

Proof:
$K_2(n)$ satisfies the recurrence

$$(n+1) \cdot K_2(n+1) - 2(2n+1) \cdot K_2(n) = 0 \quad \forall n \in \mathbb{N}$$

In view of Proposition 6, we put $f_n = \frac{K_2(n)}{4^n/\sqrt{\pi}}$. The recurrence

$$(n+1) \cdot f_{n+1} - \left(n + \frac{1}{2}\right) \cdot f_n = 0 \quad \forall n \in \mathbb{N}$$

is equivalent to

$$(X-1)\cdot f'(X)+\frac{1}{2}\cdot f(X)=0.$$

By Theorem 3(b), $b(X)$ satisfies

$$X\cdot b'(X)+\frac{1}{2}\cdot b(X)=0.$$

This implies $b(X)=\mathrm{const}X^{-\frac{1}{2}}$. From $f_n\sim n^{-\frac{1}{2}}+O\left(n^{-\frac{3}{2}}\right)$ one knows $a(T)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}T^{-\frac{1}{2}}+O\left(T^{\frac{1}{2}}\right)$ and $b(X)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}X^{-\frac{1}{2}}+O\left(X^{\frac{1}{2}}\right)$. Therefore $b(X)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}X^{-\frac{1}{2}}$. Now apply Corollary 5 (with $\frac{M}{N}=-\frac{1}{2}$, $b_0=1$, $b_n=0$ for $n>0$). We get $\mathrm{ord}_p\left(\tilde{a}_k\right)\geq 0$ for $p\neq 2$ and $\mathrm{ord}_2\left(\tilde{a}_k\right)\geq -2k-\mathrm{ord}_2\left((2k)!\right)=-3k-\mathrm{ord}_2\left(k!\right)$. Finally use the well-known estimate $\mathrm{ord}_p\left(k!\right)\leq\frac{k}{p-1}$. **q.e.d.**

5.3. Refinement for $r=3$

Proposition 9:
$K_3(n)=\frac{(3n)!}{n!^3}$ has the asymptotic expansion

$$K_3(n)\sim\frac{\sqrt{3}}{2\pi}\frac{27^n}{n}\cdot\sum_{k=0}^{\infty}\tilde{a}_k n^{-k}$$

with $\tilde{a}_k\in\mathbb{Z}\left[\frac{1}{3}\right]$ and $\mathrm{ord}_3\left(\tilde{a}_k\right)\geq -2k-\mathrm{ord}_3\left(k!\right)\geq-\frac{5}{2}k$.

Proof:
$K_3(n)$ satisfies the recurrence

$$(n+1)^2\cdot K_3(n+1)-3(3n+1)(3n+2)\cdot K_3(n)=0\quad\forall n\in\mathbb{N}$$

In view of Proposition 6, we put $f_n=\frac{K_3(n)}{\frac{\sqrt{3}}{2\pi}27^n}$. The recurrence

$$(n+1)^2\cdot f_{n+1}-\left(n+\frac{1}{3}\right)\left(n+\frac{2}{3}\right)\cdot f_n=0\quad\forall n\in\mathbb{N}\tag{4}$$

is equivalent to

$$X(X-1)\cdot f''(X)+(2X-1)\cdot f'(X)+\frac{2}{9}\cdot f(X)=0.\tag{5}$$

By Theorem 3(b), $b(X)$ satisfies

$$X(X+1)\cdot b''(X)+(2X+1)\cdot b'(X)+\frac{2}{9}\cdot b(X)=0.$$

Therefore $c(X) := b(-X)$ satisfies

$$X(X-1)\cdot c''(X) + (2X-1)\cdot c'(X) + \frac{2}{9}\cdot c(X) = 0. \tag{6}$$

From $f_n \sim \frac{1}{n}\left(1+O\left(\frac{1}{n}\right)\right)$ one knows $a(T) = 1+O(T)$, $b(X) = 1+O(X)$ and $c(X) = 1+O(X)$.

Observe that (6) is the same differential equation as (5)! Therefore the coefficients c_n of $c(X)$ satisfy the recurrence (4) and are determined by $c_0 = 1$. We conclude $c_n = \frac{K_3(n)}{27^n}$, $c(X) = \sum_{n=0}^{\infty} \frac{K_3(n)}{27^n} X^n$, $b(X) = \sum_{n=0}^{\infty} \frac{K_3(n)}{(-27)^n} X^n$. Now apply Corollary 5 (with $\frac{M}{N} = 0$ and $\operatorname{ord}_3(b_n) = -3n + \operatorname{ord}_3(K_3(n)) = -3n + \operatorname{ord}_3((3n)!) - 3\operatorname{ord}_3(n!) = -2n - 2\operatorname{ord}_3(n!)$). We get $\operatorname{ord}_p(\tilde{a}_k) \geq 0$ for $p \neq 3$ and $\operatorname{ord}_3(\tilde{a}_k) \geq \min\{-2n - \operatorname{ord}_3(n!) : 0 \leq n \leq k\} = -2k - \operatorname{ord}_3(k!)$. **q.e.d.**

5.4. The number of magic squares

Let $H_n(r)$ be the number of $n \times n$ matrices with entries in $\mathbb{N}$ and whose row and column sums each equal r. In [Hai] p. 146 it was proved that for fixed r the sequence $H_n(r)$ is P-recursive. The recurrence relation for $H_n(2)$ is well-known since [ADG]:

$$H_{n+1}(2) - (n+1)^2 \cdot H_n(2) + \frac{n^2(n+1)}{2} \cdot H_{n-1}(2) = 0 \quad \forall n \in \mathbb{N}$$

PROPOSITION 10:
$H_n(2)$ has the asymptotic expansions

(a)

$$H_n(2) \sim \sqrt{e} \cdot \frac{(2n)!}{2^{2n}} \cdot \sum_{k=0}^{\infty} a_k n^{-k}$$

with $a_k \in \mathbb{Z}\left[\frac{1}{2}\right]$, $a_0 = 1$ and $\operatorname{ord}_2(a_k) \geq -3k+1$ for $k > 0$.

(b)

$$H_n(2) \sim \sqrt{\frac{e}{\pi}} \cdot \frac{n!^2}{\sqrt{n}} \cdot \sum_{k=0}^{\infty} \tilde{a}_k n^{-k}$$

with $\tilde{a}_k \in \mathbb{Z}\left[\frac{1}{2}\right]$, $\tilde{a}_0 = 1$ and $\operatorname{ord}_2(\tilde{a}_k) \geq -3k - \operatorname{ord}_2(k!)$.

Proof:

(a) It is known ([EvSt]) that

$$H_n(2) \sim \sqrt{e} \cdot \frac{(2n)!}{2^{2n}}.$$

We put $f_n := \dfrac{H_n(2)}{\sqrt{e}\cdot\frac{(2n)!}{2^{2n}}}$. The recurrence

$$(2n+1)(2n-1)\cdot f_{n+1} - 2(n+1)(2n-1)\cdot f_n + 2n \cdot f_{n-1} = 0 \quad \forall n \in \mathbb{N}$$

is equivalent to

$$4X^2(X-1)\cdot f''(X) - 2X\big(X^2-3X-2\big)\cdot f'(X) - \big(2X^2+2X+3\big)\cdot f(X) = -3f_0.$$

By Theorem 3(b), $b(X)$ satisfies

$$4X(X+1)^2\cdot b''(X) - 2(X+1)\big(X^2-X-4\big)\cdot b'(X) - \big(2X^2+6X+7\big)\cdot b(X) = 0.$$

We substitute $b(X) = (1+X)^{\frac{1}{2}}\exp\left(\frac{X}{4}\right)\cdot c(X)$. The differential equation for $c(X)$ is simpler:

$$16X\cdot c''(X) + 32\cdot c'(X) - (X+4)\cdot c(X) = 0.$$

The coefficients c_n of $c(X)$ therefore satisfy the recurrence

$$c_{n+1} = \frac{4c_n + c_{n-1}}{16(n+1)(n+2)} \qquad \forall n\in\mathbb{N}.$$

From $f_n \sim 1 + \frac{1}{4}\frac{1}{n} + O\left(\frac{1}{n^2}\right)$ one knows $a(T) = \frac{1}{4}+O(T)$, $b(X) = \frac{1}{4}+O(X)$, $c(X) = \frac{1}{4}+O(X)$ and $c_0 = \frac{1}{4}$. The closed formula for c_n

$$c_n = \frac{1}{\lfloor\frac{n}{2}\rfloor!\lfloor\frac{n+1}{2}\rfloor!2^{3n+2}}$$

is now proved by induction, using the identity

$$\left(2\left\lfloor\frac{n+1}{2}\right\rfloor+1\right)\cdot 2\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right) = (n+1)(n+2).$$

This closed formula implies that $c_n \in \frac{1}{2^{3n+2}n!}\mathbb{Z}$. Now $b(X)$ can be written as

$$b(X) = \left(\sum_{n=0}^{\infty}\frac{\alpha_n}{2^n n!}X^n\right)\left(\sum_{n=0}^{\infty}\frac{\beta_n}{2^{2n}n!}X^n\right)\left(\sum_{n=0}^{\infty}\frac{\gamma_n}{2^{3n+2}n!}X^n\right)$$

where $\alpha_n,\beta_n,\gamma_n$ are integers. Since $\operatorname{lcm}\big\{\big(2^i i!\big)\big(2^{2j}j!\big)\big(2^{3k+2}k!\big) : i+j+k=n\big\} \mid 2^{3n+2}n!$, the coefficients b_n of $b(X)$ satisfy

$$b_n \in \frac{1}{2^{3n+2}n!}\mathbb{Z}.$$

Now Corollary 5 (with $N=1$, $M=0$) implies

$$a_{m+1} \in \frac{1}{2^{3m+2}}\mathbb{Z}.$$

(b) Combine (a) and Proposition 8, using the fact that $\operatorname{lcm}\big\{2^{3i}\cdot 2^{3j+\operatorname{ord}_2(j!)} : i+j=k\big\} \mid 2^{3k+\operatorname{ord}_2(k!)}$.

q.e.d.

5.5. The number of three-line Young tableaux

Let $S_k(n)$ be the number of Young tableaux (see [Kn] p. 61-63) with n entries and at most k rows. In [Hai] p. 166 it was proved that for fixed k the sequence $S_k(n)$ is P-recursive, and a recurrence for $S_3(n)$ was given:

$$(n+3)\cdot S_3(n+1)-(2n+3)\cdot S_3(n)-3n\cdot S_3(n-1)=0 \quad \forall n\in \mathbb{N}.$$

Proposition 11:
$S_3(n)$ has the asymptotic expansion

$$S_3(n)\sim\sqrt{\frac{27}{4\pi}}3^n n^{-\frac{3}{2}}\cdot\sum_{k=0}^{\infty}\tilde{a}_k n^{-k}$$

with $\tilde{a}_k\in\mathbb{Z}\left[\frac{1}{2}\right]$, $\tilde{a}_0=1$ and $\operatorname{ord}_2\left(\tilde{a}_k\right)\geq -4k-\operatorname{ord}_2\left(k!\right)\geq -5k$.
Proof:
It is known ([Re]) that

$$S_3(n)\sim\sqrt{\frac{27}{4\pi}}3^n n^{-\frac{3}{2}}.$$

We put $f_n:=\dfrac{S_3(n)}{\sqrt{\frac{27}{4\pi}}3^n}$. The recurrence

$$3(n+3)\cdot f_{n+1}-(2n+3)\cdot f_n-n\cdot f_{n-1}=0 \quad \forall n\in\mathbb{N}$$

is equivalent to

$$X(1-X)(3+X)\cdot f'(X)+\left(6-3X-X^2\right)\cdot f(X)=6f_0.$$

By Theorem 3(b), $b(X)$ satisfies

$$X(X+1)(X+4)\cdot b'(X)+\left(X^2+5X-2\right)\cdot b(X)=0.$$

The solution of this differential equation

$$\frac{b'(X)}{b(X)}=\frac{-X^2-5X+2}{X(X+1)(X+4)}=\frac{1/2}{X}+\frac{-2}{X+1}+\frac{1/2}{X+4}$$

is $b(X)=\text{const}X^{\frac{1}{2}}(X+1)^{-2}(X+4)^{\frac{1}{2}}$. From $f_n\sim n^{-\frac{3}{2}}+O\left(n^{-\frac{5}{2}}\right)$ one knows $a(T)=\frac{1}{\Gamma\left(\frac{3}{2}\right)}T^{\frac{1}{2}}+O\left(T^{\frac{3}{2}}\right)$ and $b(X)=\frac{1}{\Gamma\left(\frac{3}{2}\right)}X^{\frac{1}{2}}+O\left(X^{\frac{3}{2}}\right)$. Therefore

$$b(X)=\frac{1}{\Gamma\left(\frac{3}{2}\right)}X^{\frac{1}{2}}(1+X)^{-2}\left(1+\frac{X}{4}\right)^{\frac{1}{2}}.$$

Now apply Corollary 5 (with $N=2$, $M=1$). The coefficients b_n of $(1+X)^{-2}\left(1+\frac{X}{4}\right)^{\frac{1}{2}}$ satisfy $b_n \in \mathbb{Z}_p$ for $p \neq 2$ and $\operatorname{ord}_2(b_n) = -2n + \operatorname{ord}_2\left(\binom{\frac{1}{2}}{n}\right) = -3n - \operatorname{ord}_2(n!)$. Therefore $\tilde{a}_m \in \mathbb{Z}_p$ for $p \neq 2$ and

$$\begin{aligned}
\operatorname{ord}_2(\tilde{a}_m) &\geq \min\{\operatorname{ord}_2(b_n) - (2m-n) - \operatorname{ord}_2((2(m-n))!) : 0 \leq n \leq m\} \\
&= \min\{(-3n - \operatorname{ord}_2(n!)) - (2m-n) - (m-n) - \operatorname{ord}_2((m-n)!) : 0 \leq n \leq m\} \\
&= \min\{-3m - n - \operatorname{ord}_2(n!) - \operatorname{ord}_2((m-n)!) : 0 \leq n \leq m\} \\
&= -4m - \operatorname{ord}_2(m!).
\end{aligned}$$

q.e.d.

5.6. Sums of cubes of binomial coefficients

Let $S_n(r) = \displaystyle\sum_{k=0}^{n} \binom{n}{k}^r$. In [Hai] p. 175 it was proved that for fixed $r \in \mathbb{N}$ the sequence $S_n(r)$ is P-recursive, and $S_n(r) \sim 2^{rn}\left(\frac{\pi n}{2}\right)^{\frac{1-r}{2}} \frac{1}{\sqrt{r}}$. The recurrence for $S_n(3)$ was found by FRANEL in 1894:

$$n^2 \cdot S_n(3) - \left(7n^2 - 7n + 2\right) \cdot S_{n-1}(3) - 8(n-1)^2 \cdot S_{n-2}(3) = 0 \quad \forall n \in \mathbb{N}. \tag{7}$$

PROPOSITION 12:
$S_n(3)$ has the asymptotic expansion

$$S_n(3) \sim \frac{2^{3n+1}}{\sqrt{3}\pi} \cdot \frac{1}{n} \cdot \sum_{k=0}^{\infty} \tilde{a}_k n^{-k}$$

with $\tilde{a}_k \in \mathbb{Z}\left[\frac{1}{3}\right]$, $\tilde{a}_0 = 1$ and $\operatorname{ord}_3(\tilde{a}_k) \geq -2k+1$ for $k > 0$.
Proof:
Since we know that $S_n(3) \sim \frac{2^{3n+1}}{\sqrt{3}\pi n}$, we put $f_n := \dfrac{S_n(3)}{\frac{2^{3n+1}}{\sqrt{3}\pi}}$. The recurrence

$$8n^2 \cdot f_n - \left(7n^2 - 7n + 2\right) \cdot f_{n-1} - (n-1)^2 \cdot f_{n-2} = 0 \quad \forall n \in \mathbb{N}$$

is equivalent to

$$X(X-1)(X+8) \cdot f''(X) + \left(3X^2 + 14X - 8\right) \cdot f'(X) + (X+2) \cdot f(X) = 0.$$

By Theorem 3(b), $b(X)$ satisfies

$$X(X+1)(X+9) \cdot b''(X) + \left(3X^2 + 20X + 9\right) \cdot b'(X) + (X+3) \cdot b(X) = 0.$$

We first substitute $c(X) = b(-9X) \;\Leftrightarrow\; b(X) = c\left(-\frac{X}{9}\right)$ and get

$$X(X-1)(9X-1) \cdot c''(X) + \left(27X^2 - 20X + 1\right) \cdot c'(X) + 3(3X-1) \cdot c(X) = 0.$$

We now substitute $d(X)=\frac{1}{1+X}c\Big(\frac{X}{1+X}\Big) \Leftrightarrow c(X)=\frac{1}{1-X}d\Big(\frac{X}{1-X}\Big)$ and get

$$X(X+1)(8X-1)\cdot d''(X)+\big(24X^2+14X-1\big)\cdot d'(X)+2(4X+1)\cdot d(X)=0$$

which is equivalent to

$$n^2 d_n-\big(7n^2-7n+2\big)d_{n-1}-8(n-1)^2 d_{n-2}=0. \tag{8}$$

From $f_n\sim\frac{1}{n}\big(1+O\big(\frac{1}{n}\big)\big)$ one knows $a(T)=1+O(T)$, $b(X)=1+O(X)$, $c(X)=1+O(X)$, $d(X)=1+O(X)$, $d_0=1$. Observe that (8) is the same recurrence as (7)! Its solutions are determined by d_0. Therefore by induction

$$d_n=S_n(3)\in\mathbb{N}.$$

From this we get

$$c_n=\sum_{k=0}^{n}\binom{n}{k}d_k\in\mathbb{N}$$

and

$$b_n=\left(-\frac{1}{9}\right)^n c_n\in\mathbb{Z}\left[\frac{1}{3}\right].$$

We have $d_n=S_n(3)\equiv S_n(1)=2^n \pmod 3$ for all n and therefore $c_n\equiv\sum_{k=0}^{n}\binom{n}{k}2^k=3^n\equiv 0 \pmod 3$ for $n>0$, that is $\operatorname{ord}_3(b_n)\geq -2n+1$ for $n>0$. Corollary 5 (with $N=1$, $M=1$) now shows $\tilde{a}_m\in\mathbb{Z}_p$ for $p\neq 3$ and $\operatorname{ord}_3(\tilde{a}_m)\geq\min\{\operatorname{ord}_3(b_n):0\leq n\leq m\}\geq -2m+1$ for $m>0$. **q.e.d.**

5.7. The Apéry sequence for $\zeta(2)$

Let $A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$. This sequence was used in [Po] to prove that $\zeta(2)=\frac{\pi^2}{6}$ is irrational. It was shown in [Po] that the sequence A_n satisfies the recurrence

$$n^2\cdot A_n-\big(11n^2-11n+3\big)\cdot A_{n-1}-(n-1)^2\cdot A_{n-2}=0\quad\forall n\in\mathbb{N}. \tag{9}$$

From Stirling's formula, one has the asymptotic formula (see also [Po] and [Mc])

$$A_n\sim\left(\frac{1+\sqrt{5}}{2}\right)^{5n+4}\cdot\frac{1}{2\pi\sqrt{5+2\sqrt{5}}}\cdot\frac{1}{n}.$$

Proposition 13:
A_n has the asymptotic expansion

$$A_n\sim\left(\frac{1+\sqrt{5}}{2}\right)^{5n+4}\frac{1}{2\pi\sqrt{5+2\sqrt{5}}}\cdot\frac{1}{n}\cdot\sum_{k=0}^{\infty}\tilde{a}_k n^{-k}$$

with $\tilde{a}_k\in\frac{1}{\sqrt{5}^{3k}}\mathbb{Z}\Big[\frac{1+\sqrt{5}}{2}\Big]\subset\mathbb{Q}\big(\sqrt{5}\big)$ and $\tilde{a}_0=1$.

Proof:
Put $\varphi := \frac{1+\sqrt{5}}{2}$. For later use, we compute $\varphi^5 = \dfrac{11+5\sqrt{5}}{2}$ and $\varphi^{-5} = \dfrac{-11+5\sqrt{5}}{2}$. Since we know that $A_n \sim \gamma\varphi^{5n}\frac{1}{n}$ with $\gamma = \frac{\varphi^4}{2\pi\sqrt{5+2\sqrt{5}}}$, we put $f_n := \dfrac{A_n}{\gamma\varphi^{5n}}$. The recurrence

$$\varphi^5 n^2 \cdot f_n - \left(11n^2 - 11n + 3\right) \cdot f_{n-1} - \varphi^{-5}(n-1)^2 \cdot f_{n-2} = 0 \quad \forall n \in \mathbb{N}$$

is equivalent to

$$\begin{aligned} &X(X-1)\left(\varphi^{-5}X + \varphi^5\right) \cdot f''(X) + \left(3\varphi^{-5}X^2 + 22X - \varphi^5\right) \cdot f'(X) \\ &+\left(\varphi^{-5}X + 3\right) \cdot f(X) = 0. \end{aligned}$$

By Theorem 3(b), $b(X)$ satisfies

$$\begin{aligned} &X(X+1)\left(\varphi^{-5}X + 5\sqrt{5}\right) \cdot b''(X) + \left(3\varphi^{-5}X^2 + \left(-11 + 15\sqrt{5}\right)X + 5\sqrt{5}\right) \cdot b'(X) \\ &+\left(\varphi^{-5}X + \frac{-5+5\sqrt{5}}{2}\right) \cdot b(X) = 0. \end{aligned}$$

We first substitute $c(X) = b\left(-5\sqrt{5}\varphi^5 X\right) \Leftrightarrow b(X) = c\left(-\frac{X}{5\sqrt{5}\varphi^5}\right)$ and get

$$\begin{aligned} &X(X-1)\left(-5\sqrt{5}X + \varphi^{-5}\right) \cdot c''(X) + \left(-15\sqrt{5}X^2 + \left(-11 + 15\sqrt{5}\right)X - \varphi^{-5}\right) \cdot c'(X) \\ &+\left(-5\sqrt{5}X + \frac{-5+5\sqrt{5}}{2}\right) \cdot c(X) = 0. \end{aligned}$$

We now substitute $d(X) = \frac{1}{1+X}c\left(\frac{X}{1+X}\right) \Leftrightarrow c(X) = \frac{1}{1-X}d\left(\frac{X}{1-X}\right)$ and get

$$X(X+1)\left(\varphi^5 X - \varphi^{-5}\right) \cdot d''(X) + \left(3\varphi^5 X^2 + 22X - \varphi^{-5}\right) \cdot d'(X) + \left(\varphi^5 X + 3\right) \cdot d(X) = 0.$$

Next we substitute $e(X) = d\left(\varphi^{-5}X\right) \Leftrightarrow d(X) = e\left(\varphi^5 X\right)$ and get

$$X\left(X + \varphi^5\right)\left(X - \varphi^{-5}\right) \cdot e''(X) + \left(3X^2 + 22X - 1\right) \cdot e'(X) + (X+3) \cdot e(X) = 0$$

which is equivalent to

$$n^2 e_n - \left(11n^2 - 11n + 3\right)e_{n-1} - (n-1)^2 e_{n-2} = 0. \tag{10}$$

From $f_n \sim \frac{1}{n}\left(1 + O\left(\frac{1}{n}\right)\right)$ one knows $a(T) = 1 + O(T)$, $b(X) = 1 + O(X)$, $c(X) = 1 + O(X)$, $d(X) = 1 + O(X)$, $e(X) = 1 + O(X)$, $e_0 = 1$. Observe that (10) is the same recurrence as (9)! Its solutions are determined by e_0. Therefore by induction

$$e_n = A_n \in \mathbb{N}.$$

From this we get

$$d_n = \varphi^{5n} c_n \in \mathbb{Z}[\varphi],$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} d_k \in \mathbb{Z}[\varphi]$$

and

$$b_n = \left(-\frac{1}{5\sqrt{5}\varphi^5}\right)^n c_n \in \frac{1}{\sqrt{5}^{3n}} \mathbb{Z}[\varphi].$$

Since $\mathbb{Z}[\varphi]$ is a factorial ring, Corollary 5 can be generalized to hold with $\mathbb{Z}$ replaced by $\mathbb{Z}[\varphi]$ and $\mathbb{Q}$ replaced by the number field $\mathbb{Q}(\sqrt{5})$. Applying it (with $N = 1$, $M = 1$), we get $\tilde{a}_m \in \mathbb{Q}(\sqrt{5})$, $\operatorname{ord}_p(\tilde{a}_m) \geq 0$ for all primes $p \neq (\sqrt{5})$ of $\mathbb{Q}(\sqrt{5})$ and $\operatorname{ord}_{(\sqrt{5})}(\tilde{a}_m) \geq \min\left\{\operatorname{ord}_{(\sqrt{5})}(b_n) : 0 \leq n \leq m\right\} \geq -3m$. **q.e.d.**

5.8. The Apéry sequence for $\zeta(3)$

Let $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$. This sequence was used in [Po] to prove that $\zeta(3)$ is irrational. It was shown in [Po] that the sequence A_n satisfies the recurrence

$$n^3 \cdot A_n - \left(34n^3 - 51n^2 + 27n - 5\right) \cdot A_{n-1} + (n-1)^3 \cdot A_{n-2} = 0 \quad \forall n \in \mathbb{N}.$$

From Stirling's formula, one has the asymptotic formula (see also [Po] and [Mc])

$$A_n \sim \left(1 + \sqrt{2}\right)^{4n+2} \cdot \frac{1}{\left(2\pi\sqrt{2}\right)^{\frac{3}{2}}} \cdot \frac{1}{n^{\frac{3}{2}}}.$$

Proposition 14:
A_n has the asymptotic expansion

$$A_n \sim \left(1 + \sqrt{2}\right)^{4n+2} \cdot \frac{1}{\left(2\pi\sqrt{2}\right)^{\frac{3}{2}}} \cdot \frac{1}{n^{\frac{3}{2}}} \cdot \sum_{k=0}^{\infty} \tilde{a}_k n^{-k}$$

with $\tilde{a}_k \in \mathbb{Q}(\sqrt{2})$ and $\tilde{a}_0 = 1$. Under the assumption that the sequence (x_n) of rational numbers defined by $x_0 = 1$ and

$$\begin{aligned} &(n+1) \cdot x_{n+1} - 2\left(68n^2 + 68n + 27\right) \cdot x_n + 6n\left(772n^2 + 35\right) \cdot x_{n-1} \\ &-2(2n-1)^2\left(68n^2 - 68n + 27\right) \cdot x_{n-2} + (2n-1)^2(n-1)(2n-3)^2 \cdot x_{n-3} = 0 \end{aligned} \tag{11}$$

consists of integers, we have $\tilde{a}_k \in \frac{1}{2^{7k}3^k} \mathbb{Z}\left[\sqrt{2}\right]$.

Proof:
Put $\varphi := 1+\sqrt{2}$. For later use, we compute $\varphi^4 = 17+12\sqrt{2}$ and $\varphi^{-4} = 17-12\sqrt{2}$. Since we know that $A_n \sim \gamma\varphi^{4n}\frac{1}{n^{\frac{3}{2}}}$ with $\gamma = \frac{1}{\left(2\pi\sqrt{2}\right)^{\frac{3}{2}}}$, we put $f_n := \dfrac{A_n}{\gamma\varphi^{4n}}$. The recurrence

$$\varphi^4 n^3 \cdot f_n - \left(34n^3 - 51n^2 + 27n - 5\right) \cdot f_{n-1} + \varphi^{-4}(n-1)^3 \cdot f_{n-2} = 0 \quad \forall n \in \mathbb{N}$$

is equivalent to

$$\begin{aligned}&X^2(X-1)\left(\varphi^{-4}X - \varphi^4\right) \cdot f'''(X) + X\left(6\varphi^{-4}X^2 - 153X + 3\varphi^4\right) \cdot f''(X)\\ &+\left(7\varphi^{-4}X^2 - 112X + \varphi^4\right) \cdot f'(X) + \left(\varphi^{-4}X - 5\right) \cdot f(X) = 0.\end{aligned}$$

By Theorem 3(b), $b(X)$ satisfies

$$\begin{aligned}&X(X+1)^2\left(\varphi^{-4}X - 24\sqrt{2}\right) \cdot b'''(X)\\ &+(X+1)\left(6\varphi^{-4}X^2 + \left(51 - 144\sqrt{2}\right)X - 36\sqrt{2}\right) \cdot b''(X)\\ &+\left(7\varphi^{-4}X^2 + \left(126 - 168\sqrt{2}\right)X + \left(24 - 72\sqrt{2}\right)\right) \cdot b'(X)\\ &+\left(\varphi^{-4}X + \left(12 - 12\sqrt{2}\right)\right) \cdot b(X) = 0.\end{aligned}$$

We first substitute $c(X) = b\left(24\sqrt{2}\varphi^4 X\right) \;\Leftrightarrow\; b(X) = c\left(\frac{X}{24\sqrt{2}\varphi^4}\right)$ and get

$$\begin{aligned}&X(X-1)\left(24\sqrt{2}X + \varphi^{-4}\right)^2 \cdot c'''(X)\\ &+\left(24\sqrt{2}X + \varphi^{-4}\right)\left(144\sqrt{2}X^2 + \left(51 - 144\sqrt{2}\right)X - \frac{3}{2}\varphi^{-4}\right) \cdot c''(X)\\ &+\left(8064X^2 + \left(-8064 + 3024\sqrt{2}\right)X + \left(24 - 72\sqrt{2}\right)\varphi^{-4}\right) \cdot c'(X)\\ &+\left(1152X + \left(-576 + 288\sqrt{2}\right)\right) \cdot c(X) = 0.\end{aligned}$$

We now substitute $d(X) = \frac{1}{1+X}c\left(\frac{X}{1+X}\right) \;\Leftrightarrow\; c(X) = \frac{1}{1-X}d\left(\frac{X}{1-X}\right)$ and get

$$\begin{aligned}&X(X+1)^2\left(\varphi^4X + \varphi^{-4}\right)^2 \cdot d'''(X)\\ &+(X+1)\left(\varphi^4X + \varphi^{-4}\right)\left(\left(\frac{255}{2} + 90\sqrt{2}\right)X^2 + 153X + \left(\frac{51}{2} - 18\sqrt{2}\right)\right) \cdot d''(X)\\ &+\left(\left(6924 + 4896\sqrt{2}\right)X^3 + \left(8262 + 5832\sqrt{2}\right)X^2 + 3816X + \left(1326 - 936\sqrt{2}\right)\right) \cdot d'(X)\\ &+\left(\left(1731 + 1224\sqrt{2}\right)X^2 + \left(1326 + 936\sqrt{2}\right)X + 171\right) \cdot d(X) = 0.\end{aligned}$$

Next we substitute $e(X) = d\left(-\varphi^{-4}X\right) \;\Leftrightarrow\; d(X) = e\left(-\varphi^4X\right)$ and get

$$\begin{aligned}&X\left(X^2 - 34X + 1\right)^2 \cdot e'''(X) + \left(X^2 - 34X + 1\right)\left(\frac{15}{2}X^2 - 153X + \frac{3}{2}\right) \cdot e''(X)\\ &+\left(12X^3 - 486X^2 + 3816X - 78\right) \cdot e'(X) + \left(3X^2 - 78X + 171\right) \cdot e(X) = 0.\end{aligned}$$

From $f_n \sim \frac{1}{n^{\frac{3}{2}}}\left(1+O\left(\frac{1}{n}\right)\right)$ one knows $a(T)=\frac{1}{\Gamma\left(\frac{3}{2}\right)}T^{\frac{1}{2}}+O\left(T^{\frac{3}{2}}\right)$,
$b(X)=\frac{1}{\Gamma\left(\frac{3}{2}\right)}X^{\frac{1}{2}}+O\left(X^{\frac{3}{2}}\right)$, $c(X)=\frac{\sqrt{24\sqrt{2}\varphi^4}}{\Gamma\left(\frac{3}{2}\right)}X^{\frac{1}{2}}+O\left(X^{\frac{3}{2}}\right)$,
$d(X)=\frac{\sqrt{24\sqrt{2}\varphi^4}}{\Gamma\left(\frac{3}{2}\right)}X^{\frac{1}{2}}+O\left(X^{\frac{3}{2}}\right)$, $e(X)=\frac{\sqrt{-24\sqrt{2}}}{\Gamma\left(\frac{3}{2}\right)}X^{\frac{1}{2}}+O\left(X^{\frac{3}{2}}\right)$.
We therefore write $e(X)=\dfrac{\sqrt{-24\sqrt{2}}}{\Gamma\left(\frac{3}{2}\right)}\cdot\sum\limits_{n=0}^{\infty}e_nX^{n+\frac{1}{2}}$ with $e_0=1$. The differential equation for $e(X)$ is equivalent to

$$\begin{aligned}
&(n+1)(2n+1)(2n+3)\cdot e_{n+1}\\
&-2(2n+1)\left(68n^2+68n+27\right)\cdot e_n\\
&+6n\left(772n^2+35\right)\cdot e_{n-1}\\
&-2(2n-1)\left(68n^2-68n+27\right)\cdot e_{n-2}\\
&+(n-1)(2n-1)(2n-3)\cdot e_{n-3}=0 \quad \forall n\in\mathbb{N}.
\end{aligned}$$

This already shows that $e_n\in\mathbb{Q}$ for all $n\in\mathbb{N}$. If we now put $e_n=\frac{2^n n!}{(2n+1)!}\cdot x_n$, we see that the sequence (x_n) satisfies the recurrence (11). We have assumed – because we cannot prove – that $x_n\in\mathbb{Z}$. We now conclude

$$e(X)=\frac{\sqrt{-24\sqrt{2}}}{\Gamma\left(\frac{3}{2}\right)}\cdot\sum_{n=0}^{\infty}e_nX^{n+\frac{1}{2}}\quad\text{with } e_n\in\tfrac{2^n n!}{(2n+1)!}\mathbb{Z},$$

$$d(X)=\frac{\sqrt{24\sqrt{2}\varphi^4}}{\Gamma\left(\frac{3}{2}\right)}\cdot\sum_{n=0}^{\infty}d_nX^{n+\frac{1}{2}}\quad\text{with } d_n=\left(-\varphi^4\right)^n e_n\in\tfrac{2^n n!}{(2n+1)!}\mathbb{Z}\left[\sqrt{2}\right],$$

$$c(X)=\frac{\sqrt{24\sqrt{2}\varphi^4}}{\Gamma\left(\frac{3}{2}\right)}\cdot\sum_{n=0}^{\infty}c_nX^{n+\frac{1}{2}}\quad\text{with } c_n=\sum_{k=0}^{n}\binom{n+\frac{1}{2}}{n-k}d_k\in\frac{n!}{2^{\operatorname{ord}_2(n!)}(2n+1)!}\mathbb{Z}\left[\sqrt{2}\right],$$

$$b(X)=\frac{1}{\Gamma\left(\frac{3}{2}\right)}\cdot\sum_{n=0}^{\infty}b_nX^{n+\frac{1}{2}}\quad\text{with } b_n=\frac{1}{\left(24\sqrt{2}\varphi^4\right)^n}c_n\in\frac{n!}{2^{\operatorname{ord}_2(n!)+\frac{7}{2}n}3^n(2n+1)!}\mathbb{Z}\left[\sqrt{2}\right].$$

We now apply the formula contained in the proof of Corollary 5 (with $N=2$, $M=1$):

$$\tilde{a}_m=\sum_{n=0}^{m}\frac{(2n+1)!}{2^{2n}n!}\cdot b_n\cdot s_{m-n}\left(n+\frac{1}{2}\right).$$

Since $\mathbb{Z}\left[\sqrt{2}\right]$ is a factorial ring, the method of Corollary 5 can be generalized to hold with $\mathbb{Z}$ replaced by $\mathbb{Z}\left[\sqrt{2}\right]$ and $\mathbb{Q}$ replaced by the number field $\mathbb{Q}\left(\sqrt{2}\right)$. Using Lemma 4, we get

$$\operatorname{ord}_p\left(\tilde{a}_m\right)\geq\left\{\begin{array}{ll}-2\operatorname{ord}_2(m!)-11m\geq-14m & \text{for } p=\left(\sqrt{2}\right)\\ -m & \text{for } p=(3)\\ 0 & \text{for any other prime } p\end{array}\right\}$$

q.e.d.

Conjecture 1:
The x_n defined above are indeed integers.
Conjecture 2:
$\operatorname{ord}_3\left(\tilde{a}_k\right)\geq 0$.

5.9. The number of Baxter permutations

Let $B(n)$ be the number of Baxter permutations, see [CGHK]:

$$B(n) = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \cdot \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

In [Hai] p. 173 we proved the following recurrence:

$$\begin{aligned}&(n+1)(n+2)\cdot B(n-1) - \left(7n^2-7n-2\right)\cdot B(n-2)\\ &-8(n-2)(n-3)\cdot B(n-3) = 0 \quad \forall n \geq 3\end{aligned} \tag{12}$$

Proposition 15:
$B(n)$ has the asymptotic expansion

$$B(n) \sim \frac{32}{\sqrt{3}\pi}\frac{8^n}{n^4} \cdot \sum_{k=0}^{\infty} \tilde{a}_k n^{-k}$$

with $\tilde{a}_k \in \mathbb{Z}\left[\frac{1}{3}\right]$, $\tilde{a}_0 = 1$ and $\operatorname{ord}_3\left(\tilde{a}_k\right) \geq -2k$.
Proof:
From Stirling's formula it follows that

$$B(n) \sim \frac{32}{\sqrt{3}\pi}\frac{8^n}{n^4}.$$

We put $f_n := \dfrac{B(n)}{\frac{32}{\sqrt{3}\pi}8^n}$. The recurrence

$$8(n+1)(n+2)\cdot f_{n-1} - \left(7n^2-7n-2\right)\cdot f_{n-2} - (n-2)(n-3)\cdot f_{n-3} = 0 \quad \forall n \geq 3$$

is equivalent to

$$\begin{aligned}&X^2(X+8)(X-1)\cdot f''(X) + 2X\left(X^2+14X-24\right)\cdot f'(X) + 12(X-4)\cdot f(X) =\\ &= (12f_0 - 96f_1)X - 48f_0.\end{aligned}$$

By Theorem 3(b), $b(X)$ satisfies

$$X(X+1)^2(X+9)\cdot b''(X) + 2(X+1)\left(X^2+16X-9\right)\cdot b'(X) + 12(X-3)\cdot b(X) = 0.$$

We first substitute $c(X) = \frac{6}{(-9)^3}b(-9X) \Leftrightarrow b(X) = \frac{(-9)^3}{6}c\left(-\frac{X}{9}\right)$ and get

$$\begin{aligned}&X(X-1)(9X-1)^2\cdot c''(X) + 2(9X-1)\left(9X^2-16X-1\right)\cdot c'(X)\\ &-36(3X+1)\cdot c(X) = 0.\end{aligned}$$

Next we substitute $d(X) = \frac{1}{1+X} c\left(\frac{X}{1+X}\right) \Leftrightarrow c(X) = \frac{1}{1-X} d\left(\frac{X}{1-X}\right)$ and get

$$X(X+1)(8X-1)^2 \cdot d''(X) + 2(8X-1)\left(24X^2+16X+1\right) \cdot d'(X)$$
$$+2\left(128X^2+64X+17\right) \cdot d(X) = 0.$$

We now substitute $e(X) = (8X-1)^2 d(X) \Leftrightarrow d(X) = (8X-1)^{-2} e(X)$ and get

$$X(X+1)(8X-1) \cdot e''(X) + \left(16X^2+2\right) \cdot e'(X) - (16X+2) \cdot e(X) = 0.$$

Therefore the coefficients e_n of $e(X)$ satisfy the recurrence

$$(n+1)(n-2) \cdot e_{n+1} - \left(7n^2-7n-2\right) \cdot e_n - 8(n+1)(n-2) \cdot e_{n-1} = 0 \quad \forall n \geq 3 \tag{13}$$

From $f_n \sim \frac{1}{n^4}\left(1+O\left(\frac{1}{n}\right)\right)$ one knows $a(T) = \frac{1}{6}T^3 + O\left(T^4\right)$, $b(X) = \frac{1}{6}X^3 + O\left(X^4\right)$, $c(X) = X^3 + O\left(X^4\right)$, $d(X) = X^3 + O\left(X^4\right)$, $e(X) = X^3 + O\left(X^4\right)$, that is $e_0 = 0$, $e_1 = 0$, $e_2 = 0$, $e_3 = 1$. Comparing (12) and (13), we see that $e'_n := \binom{n+1}{4} B(n-2)$ also satisfies (13), with the same initial conditions. Therefore by induction

$$e_n = \binom{n+1}{4} B(n-2) \in \mathbb{N}.$$

From this we get $e(X) \in \mathbb{Z}[\![X]\!]$, $d(X) \in \mathbb{Z}[\![X]\!]$, $c(X) \in \mathbb{Z}[\![X]\!]$ and
$3! \cdot b(X) = X^3 \cdot \sum_{n=0}^{\infty} b_n X^n$ with $b_n \in \frac{1}{3^{2n}}\mathbb{Z}$. Corollary 5 (with $N = 1$, $M = 3$) now shows $\tilde{a}_m \in \mathbb{Z}_p$ for $p \neq 3$ and $\operatorname{ord}_3(\tilde{a}_m) \geq \min\{\operatorname{ord}_3(b_n) : 0 \leq n \leq m\} \geq -2m$. **q.e.d.**

5.10. The number of vertices of certain polytopes

Let M_n be the number of vertices of the polytope of symmetric doubly stochastic $n \times n$ matrices. Put $m_n := \frac{M_n}{n!}$. It was proved in [Ka1] and [Hai] p. 163 that the generating function of this sequence is

$$m(X) = \sum_{n=0}^{\infty} m_n X^n = \left(\frac{1+X}{1-X}\right)^{1/4} \exp\left(\frac{X}{2} + \frac{X^2}{2}\right). \tag{14}$$

One therefore has the differential equation

$$P_1(X) m'(X) + P_0(X) m(X) = 0 \tag{15}$$

with $P_1(X) = 2\left(1-X^2\right)$ and $P_0(X) = -2 - 2X + X^2 + 2X^3$ which is equivalent to the recurrence

$$2(n+1) \cdot m_{n+1} - 2 \cdot m_n - 2n \cdot m_{n-1} + m_{n-2} + 2 \cdot m_{n-3} = 0. \tag{16}$$

PROPOSITION 16:
m_n has the asymptotic expansion

$$m_n \sim n^{-\frac{3}{4}} \cdot c^{(1)} \cdot \left(\tilde{a}_0^{(1)} + \frac{\tilde{a}_1^{(1)}}{n} + \cdots\right) + (-1)^n n^{-\frac{5}{4}} \cdot c^{(2)} \cdot \left(\tilde{a}_0^{(2)} + \frac{\tilde{a}_1^{(2)}}{n} + \cdots\right) \quad (17)$$

with $c^{(1)} = \frac{2^{1/4}e}{\Gamma(\frac{1}{4})}$, $c^{(2)} = \frac{2^{-1/4}}{\Gamma(-\frac{1}{4})}$,
all $\tilde{a}_k^{(j)} \in \mathbb{Z}\left[\frac{1}{2}\right]$, $\tilde{a}_0^{(j)} = 1$ and $\mathrm{ord}_2\left(\tilde{a}_k^{(j)}\right) \geq -5k - \mathrm{ord}_2(k!) \geq -6k$.

Proof:
Applying the methods of [FlOd1] Prop.6 or [FlOd2] Cor.3 to (14), we get an asymptotic expansion (17) with $c^{(1)} = \frac{2^{1/4}e}{\Gamma(\frac{1}{4})}$, $c^{(2)} = \frac{2^{-1/4}}{\Gamma(-\frac{1}{4})}$ and $\tilde{a}_0^{(j)} = 1$. By Proposition 1 there are sequences $\left(m_n^{(1)}\right)$ and $\left(m_n^{(2)}\right)$ that satisfy the recurrence (16), have the prescribed asymptotic behaviour

$$m_n^{(1)} \sim n^{-\frac{3}{4}} \cdot \left(\tilde{a}_0^{(1)} + \frac{\tilde{a}_1^{(1)}}{n} + \cdots\right)$$

$$m_n^{(2)} \sim (-1)^n n^{-\frac{5}{4}} \cdot \left(\tilde{a}_0^{(2)} + \frac{\tilde{a}_1^{(2)}}{n} + \cdots\right)$$

and together make up m_n:

$$m_n = c^{(1)} m_n^{(1)} + c^{(2)} m_n^{(2)}.$$

We now examine $m_n^{(1)}$ and $m_n^{(2)}$ separately.
First put $f_n = m_n^{(1)}$. By (16), its generating function $f(X)$ satisfies a differential equation $P_1(X)f'(X) + P_0(X)f(X) \in \mathbb{C}[X]$. By Theorem 3(b), $b(X)$ satisfies

$$P_1(X+1)b'(X) + P_0(X+1)b(X) = 0.$$

Since this differential equation is linear of order 1, a comparison with (15) yields $b(X) = \mathrm{const} \cdot m(X+1)$. From $\tilde{a}_0^{(1)} = 1$ one knows $a(T) = \frac{1}{\Gamma(\frac{3}{4})}T^{-\frac{1}{4}} + O\left(T^{\frac{3}{4}}\right)$ and $b(X) = \frac{1}{\Gamma(\frac{3}{4})}X^{-\frac{1}{4}} + O\left(X^{\frac{3}{4}}\right)$. Therefore

$$b(X) = \frac{1}{\Gamma\left(\frac{3}{4}\right)}X^{-\frac{1}{4}} \cdot \left(1 + \frac{X}{2}\right)^{1/4} \exp\left(\frac{3}{2}X + \frac{X^2}{2}\right).$$

From $\left(1+\frac{X}{2}\right)^{1/4} = \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{3n} n!} X^n$ with $\alpha_n \in \mathbb{Z}$ and $\exp\left(\frac{3}{2}X + \frac{X^2}{2}\right) = \sum_{n=0}^{\infty} \frac{\beta_n}{2^n n!} X^n$ with $\beta_n \in \mathbb{Z}$, we infer that we can apply Corollary 5 with $N = 4$, $M = -1$, $b_n \in \frac{1}{2^{3n} n!}\mathbb{Z}$.

We get $\operatorname{ord}_p\left(\tilde{a}_m^{(1)}\right) \geq 0$ for $p \neq 2$ and

$$\begin{aligned}\operatorname{ord}_2\left(\tilde{a}_m^{(1)}\right) &\geq \min\{(-3n - \operatorname{ord}_2(n!)) - 2(2m-n) - \operatorname{ord}_2((2(m-n))!) : 0 \leq n \leq m\} \\ &= -5m - \operatorname{ord}_2(m!).\end{aligned}$$

Next put $f_n = (-1)^n m_n^{(1)}$. By (16), its generating function $f(X)$ satisfies a differential equation $P_1(-X)f'(X) - P_0(-X)f(X) \in \mathbb{C}[X]$. By Theorem 3(b), $b(X)$ satisfies

$$P_1(-X-1)b'(X) - P_0(-X-1)b(X) = 0.$$

Since this differential equation is linear of order 1, a comparison with (15) yields $b(X) = \text{const} \cdot m(-X-1)$. From $\tilde{a}_0^{(2)} = 1$ one knows $a(T) = \frac{1}{\Gamma\left(\frac{5}{4}\right)}T^{\frac{1}{4}} + O\left(T^{\frac{5}{4}}\right)$ and $b(X) = \frac{1}{\Gamma\left(\frac{5}{4}\right)}X^{\frac{1}{4}} + O\left(X^{\frac{5}{4}}\right)$. Therefore

$$b(X) = \frac{1}{\Gamma\left(\frac{5}{4}\right)}X^{\frac{1}{4}} \cdot \left(1 + \frac{X}{2}\right)^{-1/4} \exp\left(\frac{X}{2} + \frac{X^2}{2}\right).$$

From $\left(1 + \frac{X}{2}\right)^{-1/4} = \sum_{n=0}^{\infty} \frac{\alpha_n}{2^{3n} n!} X^n$ with $\alpha_n \in \mathbb{Z}$ and $\exp\left(\frac{X}{2} + \frac{X^2}{2}\right) = \sum_{n=0}^{\infty} \frac{\beta_n}{2^n n!} X^n$ with $\beta_n \in \mathbb{Z}$, we infer that we can apply Corollary 5 with $N = 4$, $M = 1$, $b_n \in \frac{1}{2^{3n} n!}\mathbb{Z}$. Just as above, we get $\operatorname{ord}_p\left(\tilde{a}_m^{(2)}\right) \geq 0$ for $p \neq 2$ and $\operatorname{ord}_2\left(\tilde{a}_m^{(2)}\right) \geq -5m - \operatorname{ord}_2(m!)$. **q.e.d.**

As another example, let M_n be the number of vertices of the polytope of symmetric doubly substochastic $n \times n$ matrices. Put $m_n := \frac{M_n}{n!}$. It was proved in [Ka2] and [Hai] p. 164 that the generating function of this sequence is

$$m(X) = \sum_{n=0}^{\infty} m_n X^n = \left(\frac{1+X}{1-X}\right)^{1/4} \exp\left(\frac{3}{2}X + \frac{X^2}{2}\right).$$

One therefore has the differential equation

$$P_1(X)m'(X) + P_0(X)m(X) = 0$$

with $P_1(X) = 2\left(1 - X^2\right)$ and $P_0(X) = -4 - 2X + 3X^2 + 2X^3$ which is equivalent to the recurrence

$$2(n+1) \cdot m_{n+1} - 4 \cdot m_n - 2n \cdot m_{n-1} + 3m_{n-2} + 2 \cdot m_{n-3} = 0.$$

Proposition 17:

m_n has the asymptotic expansion

$$m_n \sim n^{-\frac{3}{4}} \cdot c^{(1)} \cdot \left(\tilde{a}_0^{(1)} + \frac{\tilde{a}_1^{(1)}}{n} + \cdots\right) + (-1)^n n^{-\frac{5}{4}} \cdot c^{(2)} \cdot \left(\tilde{a}_0^{(2)} + \frac{\tilde{a}_1^{(2)}}{n} + \cdots\right)$$

with $c^{(1)} = \frac{2^{1/4}e^2}{\Gamma\left(\frac{1}{4}\right)}$, $c^{(2)} = \frac{2^{-1/4}e^{-1}}{\Gamma\left(-\frac{1}{4}\right)}$,

all $\tilde{a}_k^{(j)} \in \mathbb{Z}\left[\frac{1}{2}\right]$, $\tilde{a}_0^{(j)} = 1$ and $\operatorname{ord}_2\left(\tilde{a}_k^{(j)}\right) \geq -5k - \operatorname{ord}_2(k!) \geq -6k$.

Proof:
Literally the same as the proof of Proposition 16, except that the equations have to be slightly modified:

$$b(X) = \frac{1}{\Gamma\left(\frac{3}{4}\right)} X^{-\frac{1}{4}} \cdot \left(1 + \frac{X}{2}\right)^{1/4} \exp\left(\frac{5}{2}X + \frac{X^2}{2}\right)$$

and

$$b(X) = \frac{1}{\Gamma\left(\frac{5}{4}\right)} X^{\frac{1}{4}} \cdot \left(1 + \frac{X}{2}\right)^{-1/4} \exp\left(-\frac{X}{2} + \frac{X^2}{2}\right),$$

respectively. **q.e.d.**

6. Behind the Examples

We now discuss the examples and the methods used there. Apart from 5.1. all the results were based on the ability to give a closed formula for $b(X)$ or its coefficients b_n. In 5.2., 5.5. and 5.10. this was easily done because the differential equation for $b(X)$ was of order 1. In 5.4. we needed some more luck.
In 5.3., 5.6., 5.7. and 5.9. it was possible to relate the sequence (b_n) to the original sequence and to exploit that the latter was a priori known to assume integral values. Let us explain the genereal idea behind this.
If a D-finite function $f(X)$ satisfies a differential equation $\sum_{k=0}^{r} p_k(X) f^{(k)}(X) = 0$ with polynomials $p_k \in \mathbb{C}[X]$ and $p_r(X) = (X - \alpha_1) \cdots (X - \alpha_s)$, then $\alpha_1, \ldots, \alpha_s$ (and possibly ∞) are called the *singular points* of this differential equation or – sloppily – of f.
We start from a sequence (F_n) in $\mathbb{Z}$. Call $\rho = \lim\limits_{n\to\infty} F_n{}^{\frac{1}{n}}$ its growth factor. It is well-known that $\frac{1}{|\rho|}$ is precisely the convergence radius of its generating function $F(X)$ viewed as a complex analytic function. In other words, $\frac{1}{|\rho|} = \min\{|\alpha_j| : \alpha_j \neq 0\}$.
We then put $f_n = \frac{F_n}{\gamma_0 \rho^n}$ with some constant $\gamma_0 \neq 0$. Its generating function is simply $f(X) = \frac{1}{\gamma_0} F\left(\frac{X}{\rho}\right)$. Its singular points are $\rho\alpha_1, \ldots, \rho\alpha_s, \infty$. Since $\lim\limits_{n\to\infty} f_n{}^{\frac{1}{n}} = 1$, 1 is among them.
By Theorem 3(b), the singular points of $b(X)$ are $\rho\alpha_1 - 1, \ldots, \rho\alpha_s - 1, \infty$, and 0 is among them. We now use a couple of linear fractional (Möbius) transformations to map these singular points onto those of f or F. These Möbius transformations must of course map 0 to 0 since we are handling power series developed around 0 and want to get information about its coefficients.
Example 5.3. went like this: Start with a recurrence

$$(n+1)\ (\alpha n + \beta) f_{n+1} - \left(\alpha n^2 + (2\beta - \alpha) n + \gamma\right) f_n = 0 \quad \forall n \in \mathbb{N} \qquad (*)$$

with $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha \neq 0$, $\gamma \neq 0$, $-\frac{\beta}{\alpha} \notin \mathbb{N}$. (Example 5.3. was the special case $\alpha = 1$, $\beta = 1$, $\gamma = \frac{2}{9}$.) This is equivalent to

$$\alpha X(X-1) \cdot f''(X) + \beta(2X - 1) \cdot f'(X) + \gamma \cdot f(X) = 0.$$

These are the singular points:

$f(X)$	0	1	∞
$b(X)$	-1	0	∞
$c(X)$	1	0	∞

To achieve this, we simply set $c(X) := b(-X)$. The differential equations for $b(X)$ and $c(X)$:

$$\alpha X(X+1)\cdot b''(X)+\beta(2X+1)\cdot b'(X)+\gamma\cdot b(X)=0,$$
$$\alpha X(X-1)\cdot c''(X)+\beta(2X-1)\cdot c'(X)+\gamma\cdot c(X)=0.$$

Assuming that $c(X)\in\mathbb{C}[\![X]\!]$, the coefficients of $c(X)$ satisfy the same recurrence relation $(*)$ and are thus determined by the first coefficient. We conclude that $c(X)$ and $f(X)$ are linearly dependent.
Examples 5.6. and 5.7. went like this: Start with a recurrence

$$\alpha n^2 F_n+\left(\beta\left(n-\frac{1}{2}\right)^2+\beta'\right)F_{n-1}+\gamma(n-1)^2F_{n-2}=0\quad\forall n\in\mathbb{Z}\tag{$*$}$$

with $\alpha,\beta,\beta',\gamma\in\mathbb{C}$, $\alpha\neq 0$, $\gamma\neq 0$, $\beta^2-4\alpha\gamma>0$. (Example 5.6. was the special case $\alpha=1$, $\beta=-7$, $\beta'=-\frac{1}{4}$, $\gamma=-8$, and Example 5.7. was the special case $\alpha=1$, $\beta=-11$, $\beta'=-\frac{1}{4}$, $\gamma=-1$.) This is equivalent to

$$X\left(\alpha+\beta X+\gamma X^2\right)\cdot F''(X)+\left(\alpha+2\beta X+3\gamma X^2\right)\cdot F'(X)+\left(\frac{\beta}{4}+\beta'+\gamma X\right)\cdot F(X)=0.$$

Let φ be the solution of the quadratic equation $\alpha x^2+\beta x+\gamma=0$ with largest absolute value. The growth factor of F is $\rho=\varphi$. We therefore substitute $F(X)=\gamma_0 f(\varphi X)\iff f(X)=\frac{1}{\gamma_0}F\left(\frac{X}{\varphi}\right)$. These are the singular points:

$F(X)$	0	$\frac{1}{\varphi}$	$\frac{\alpha\varphi}{\gamma}$	∞
$f(X)$	0	1	$\frac{\alpha\varphi^2}{\gamma}$	∞
$b(X)$	-1	0	$\frac{\alpha\varphi^2}{\gamma}-1$	∞

We are looking for a Möbius transformation that maps the singular points of $b(X)$ to the singular points of $F(X)$, mapping 0 to 0. This is indeed possible since they have the same cross ratio: $\lambda\left(-1,0,\frac{\alpha\varphi^2}{\gamma}-1,\infty\right)=\lambda\left(\frac{1}{\varphi},0,\infty,\frac{\alpha\varphi}{\gamma}\right)$. We write this Möbius transformation as composition of three elementary substitutions:

$$c(X)=b\left(\left(\frac{\alpha\varphi^2}{\gamma}-1\right)X\right)\iff b(X)=c\left(\frac{X}{\frac{\alpha\varphi^2}{\gamma}-1}\right)$$

$$d(X)=\frac{1}{1+X}c\left(\frac{X}{1+X}\right)\iff c(X)=\frac{1}{1-X}d\left(\frac{X}{1-X}\right)$$

$$e(X) = d\Big(-\frac{\gamma}{\alpha\varphi}X\Big) \;\Leftrightarrow\; d(X) = e\Big(-\frac{\alpha\varphi}{\gamma}X\Big)$$

The singular points are transformed like this:

$b(X)$	-1	0	$\frac{\alpha\varphi^2}{\gamma} - 1$	∞
$c(X)$	$-\frac{1}{\frac{\alpha\varphi^2}{\gamma}-1}$	0	1	∞
$d(X)$	$-\frac{\gamma}{\alpha\varphi^2}$	0	∞	-1
$e(X)$	$\frac{1}{\varphi}$	0	∞	$\frac{\alpha\varphi}{\gamma}$

We then expect to find some relation between $e(X)$ and $F(X)$. In fact, after a long and tedious computation – better done with Maple V than by hand – one gets

$$X\left(\alpha + \beta X + \gamma X^2\right) \cdot e''(X) + \left(\alpha + 2\beta X + 3\gamma X^2\right) \cdot e'(X) + \left(\frac{\beta}{4} + \beta' + \gamma X\right) \cdot e(X) = 0.$$

Assuming that $e(X) \in \mathbb{C}[\![X]\!]$, the coefficients of $e(X)$ satisfy the same recurrence relation $(*)$ and are thus determined by the first coefficient. We conclude that $e(X)$ and $F(X)$ are linearly dependent.
Example 5.9. was done similarly, with the only modification that the singular points don't all have multiplicity one.
Example 5.8. was done similarly: Start with a recurrence

$$\alpha n^3 F_n + \left(\beta\left(n - \frac{1}{2}\right)^3 + \beta'\left(n - \frac{1}{2}\right)\right) F_{n-1} + \gamma(n-1)^3 F_{n-2} = 0 \quad \forall n \in \mathbb{Z}$$

with $\alpha, \beta, \beta', \gamma \in \mathbb{C}$, $\alpha \neq 0$, $\gamma \neq 0$, $\beta^2 - 4\alpha\gamma > 0$. (Example 5.8. was the special case $\alpha = 1$, $\beta = -34$, $\beta' = -\frac{3}{2}$, $\gamma = 1$.) This is equivalent to

$$\begin{aligned} &X^2\left(\alpha + \beta X + \gamma X^2\right) \cdot F'''(X) + X\left(3\alpha + \frac{9}{2}\beta X + 6\gamma X^2\right) \cdot F''(X) \\ &+ \left(\alpha + \left(\frac{13}{4}\beta + \beta'\right)X + 7\gamma X^2\right) \cdot F'(X) + \left(\frac{\beta}{8} + \frac{\beta'}{2} + \gamma X\right) \cdot F(X) = 0. \end{aligned}$$

We make the same substitutions as above, getting $f(X)$, $b(X)$, $c(X)$, $d(X)$, $e(X)$. After a very long computation, one gets the following differential equation for $e(X)$:

$$\begin{aligned} &X\left(\alpha + \beta X + \gamma X^2\right)^2 \cdot e'''(X) + \frac{3}{2}\left(\alpha + \beta X + \gamma X^2\right)\left(\alpha + 3\beta X + 5\gamma X^2\right) \cdot e''(X) \\ &+ \left(\alpha\left(\frac{9}{4}\beta + \beta'\right) + \left(\beta\beta' + \frac{13}{4}\beta^2 + 8\alpha\gamma\right)X + \gamma\left(\frac{57}{4}\beta + \beta'\right)X^2 + 12\gamma^2 X^3\right) \cdot e'(X) \\ &+ \left(\left(\frac{1}{8}\beta^2 + \frac{1}{2}\beta\beta' + \alpha\gamma\right) + \gamma\left(\frac{9}{4}\beta + \beta'\right)X + 3\gamma^2 X^2\right) \cdot e(X) = 0. \end{aligned}$$

Writing $e(X) = \gamma_1 \sum_{n=0}^{\infty} e_n X^{n+\frac{1}{2}}$ with some constant $\gamma_1 \neq 0$, this is equivalent to the recurrence

$$\begin{aligned}
&\alpha^2(n+1)(2n+1)(2n+3)\cdot e_{n+1}\\
&+\alpha(2n+1)\left(\beta\left((2n+1)^2+\frac{1}{2}\right)+2\beta'\right)\cdot e_n\\
&+2n\big(\alpha\gamma\big(4n^2+3\big)+2\beta^2n^2+2\beta\beta'\big)\cdot e_{n-1}\\
&+\gamma(2n-1)\left(\beta\left((2n-1)^2+\frac{1}{2}\right)+2\beta'\right)\cdot e_{n-2}\\
&+\gamma^2(n-1)(2n-1)(2n-3)\cdot e_{n-3}=0 \quad \forall n\in\mathbb{N}.
\end{aligned}$$

It is not clear to us how this sequence (e_n) is related to (F_n).

7. References

[ADG] Harsh Anand, Vishwa Chander Dumir, Hansraj Gupta: A combinatorial distribution problem.
Duke Math. J. **33** (1966), 757-769.

[Bi] George D. Birkhoff: Formal theory of irregular difference equations.
Acta Math. **54** (1930), 205-246.

[BiTr] George D. Birkhoff, W. J. Trjitzinsky: Analytic theory of singular difference equations.
Acta Math. **60** (1932), 1-89.

[CGHK] Fan R. K. Chung, Ronald L. Graham, V. E. Hogatt Jr., M. Kleiman: The number of Baxter permutations.
J. Combinatorial Theory (A) **24** (1978), 382-394.

[EvSt] C. J. Everett, P. R. Stein: The asymptotic number of integer stochastic matrices.
Discrete Math. **1** (1971), 55-72.

[FlOd1] Philippe Flajolet, Andrew Odlyzko: The average height of binary trees and other simple trees.
Journal of Computer and System Sciences **25** (1982), 171-213.

[FlOd2] Philippe Flajolet, Andrew Odlyzko: Singularity analysis of generating functions.
SIAM J. Discrete Math. **3** (1990), 216-240.

[Hai] Bruno Haible: D-finite Potenzreihen in mehreren Variablen.
Diplomarbeit, Universität Karlsruhe, June 1989.

[HalHei] Heinz-Richard Halder, Werner Heise: Einführung in die Kombinatorik.
Carl Hanser Verlag. München, Wien 1976.

[Ka1] M. Katz: On the extreme points of a certain convex polytope.
J. Combinatorial Theory **8** (1970), 417-423.

[Ka2] M. Katz: On the extreme points of the set of substochastic and symmetric matrices.
J. Math. Anal. Appl. **37** (1972), 576-579.

[Kn] Donald E. Knuth: The Art of Computer Programming. Vol. 3: Sorting and Searching.
Addison-Wesley, Reading, Mass. 1968.

[Mc] Richard J. McIntosh: An asymptotic formula for binomial sums.
Preprint.

[Po] Alfred van der Poorten: A proof that Euler missed ..., Apéry's proof of the irrationality of $\zeta(3)$.
Math. Intelligencer **1** (1979), 195-203.

[Re] Amitai Regev: Asymptotic values for degrees associated with strips of Young diagrams.
Advances in Math. **41** (1981), 115-136.

[Sta] Richard P. Stanley: Differentiably finite power series.
European J. Combin. **1** (1980), 175-188. MR # 81m:05012.

[WiZe] Jet Wimp, Doron Zeilberger: Resurrecting the asymptotics of linear recurrences.
Journal of Mathematical Analysis and Applications **111** (1985), 162-176. MR # 87b:05015.