Combinatorial proof of a binomial identity

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Abstract: We prove that $\sum_{k=0}^{n}(3k-2n)\cdot\binom{n}{k}^2\cdot\binom{2k}{k}=0$ for $n\in\mathbb{N}_0$.

In May 1990 D. Zeilberger asked for a combinatorial proof of

$$\forall n \in \mathbb{N}_0 : \sum_{k=0}^{n}(3k-2n)\cdot\binom{n}{k}^2\cdot\binom{2k}{k}=0 \tag{1}$$

We will prove it in the form

$$\sum_{k=0}^{n} k\cdot\binom{n}{k}^2\cdot\binom{2k}{k} = 2\cdot\sum_{k=0}^{n}(n-k)\cdot\binom{n}{k}^2\cdot\binom{2k}{k} \tag{2}$$

First proof: The generating function $f(x) := \sum_{n=0}^{\infty} \frac{1}{n!^2} \left(\sum_{k=0}^{n} \binom{n}{k}^2 \cdot \binom{2k}{k} \right) x^n$ and some modifications of it will give the result. First observe that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!^2} \frac{(2k)!}{k!^4} x^n \\ &= \sum_{j,k=0}^{\infty} \frac{1}{j!^2} \frac{(2k)!}{k!^4} x^{j+k} \\ &= \underbrace{\sum_{j=0}^{\infty} \frac{1}{j!^2} x^j}_{=:\, g(x)} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{(2k)!}{k!^4} x^k}_{=:\, h(x)} \end{aligned}$$

The clou of all this is the observation

$$\sum_{j=0}^{k} \frac{1}{j!^2} \frac{1}{(k-j)!^2} = \frac{1}{k!^2} \sum_{j=0}^{k} \binom{k}{j} \binom{k}{k-j} = \frac{1}{k!^2} \binom{2k}{k} = \frac{(2k)!}{k!^4}$$

that is

$$g(x)^2 = h(x) \tag{3}$$

Take the derivative of it, to get

$$g(x) \cdot h'(x) = 2 \cdot g'(x) \cdot h(x) \tag{4}$$

Now

$$\sum_{n=0}^{\infty} \frac{1}{n!^2} \left(\sum_{k=0}^{n} k \cdot \binom{n}{k}^2 \cdot \binom{2k}{k} \right) x^n = \sum_{j,k=0}^{\infty} \frac{1}{j!^2} k \frac{(2k)!}{k!^4} x^{j+k} = g(x) \cdot x h'(x)$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n!^2} \left(\sum_{k=0}^{n} (n-k) \cdot \binom{n}{k}^2 \cdot \binom{2k}{k} \right) x^n = \sum_{j,k=0}^{\infty} \frac{j}{j!^2} \frac{(2k)!}{k!^4} x^{j+k} = x g'(x) \cdot h(x)$$

Using (4), we get (2).

Second proof: This proof contains no words about generating functions. It is based on

$$\sum_{j=0}^{k} \binom{k}{j}^2 = \binom{2k}{k} \tag{5}$$

We have

$$\begin{aligned}
\sum_{k=0}^{n} k \cdot \binom{n}{k}^2 \cdot \binom{2k}{k} &= \sum_{k=0}^{n} k \cdot \binom{n}{k}^2 \cdot \sum_{j=0}^{k} \binom{k}{j}^2 && \text{by (5)} \\
&= \sum_{0 \le j \le k \le n} k \cdot \left(\binom{n}{k} \cdot \binom{k}{j} \right)^2 \\
&= \sum_{0 \le j \le k \le n} j \cdot \left(\binom{n}{k} \cdot \binom{k}{j} \right)^2 + \sum_{0 \le j \le k \le n} (k-j) \cdot \left(\binom{n}{k} \cdot \binom{k}{j} \right)^2 \\
& && \text{splitting up } k = j + (k-j) \\
&= 2 \cdot \sum_{0 \le j \le k \le n} j \cdot \left(\binom{n}{k} \cdot \binom{k}{j} \right)^2 \\
& && \text{by transformation of the second sum substituting } j \mapsto k-j \\
&= 2 \cdot \sum_{0 \le j \le k \le n} j \cdot \left(\binom{n}{j} \cdot \binom{n-j}{k-j} \right)^2 \\
&= 2 \cdot \sum_{j=0}^{n} j \cdot \binom{n}{j}^2 \cdot \sum_{k=j}^{n} \binom{n-j}{k-j}^2 \\
&= 2 \cdot \sum_{j=0}^{n} j \cdot \binom{n}{j}^2 \cdot \binom{2(n-j)}{n-j} && \text{by (5)} \\
&= 2 \cdot \sum_{k=0}^{n} (n-k) \cdot \binom{n}{k}^2 \cdot \binom{2k}{k} && \text{by substitution } j = n-k
\end{aligned}$$

Third proof: Let us write $X \underset{k}{\subset} Y$ for $X \subset k \wedge \#X = k$. Let two disjoint n-element-sets A and B be given. Let us count

$$N := \#\{(C, D, E, x) \mid C \subset A, D \subset B, x \in E \subset C \dot\cup D, \#E = \#C = \#D\}$$

On one hand, denoting $k = \#E = \#C = \#D$, this is

$$N = \sum_{k=0}^{n} \sum_{C \underset{k}{\subset} A} \sum_{D \underset{k}{\subset} B} \sum_{E \underset{k}{\subset} C \dot\cup D} \sum_{x \in E} 1 = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{k} \cdot \binom{2k}{k} \cdot k$$

On the other hand, we may restrict ourselves to the case $x \in E \cap A = E \cap C$, since by symmetry ($A \leftrightarrow B$, $C \leftrightarrow D$) the case $x \in E \cap B = E \cap D$ will give the same number once more.

$$N = 2 \cdot \#\{(C, D, E, x) \mid C \subset A, D \subset B, E \subset C \dot\cup D, \#E = \#C = \#D, x \in E \cap A\}$$

Putting $A_1 := A \setminus C$, $A_2 := C \setminus E$, $A_3 := E \cap A$ and $B_1 := B \setminus D$, $B_2 := D \setminus E$, $B_3 := E \cap B$, we get a bijection:

$$\begin{aligned}
N &= 2 \cdot \#\{(A_1, A_2, A_3, B_1, B_2, B_3) \mid A_1 \dot\cup A_2 \dot\cup A_3 = A, B_1 \dot\cup B_2 \dot\cup B_3 = B, x \in A_3, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \#(A_3 \dot\cup B_3) = \#(A_2 \dot\cup A_3) = \#(B_2 \dot\cup B_3)\} \\
&= 2 \cdot \#\{(A_1, A_2, A_3, B_1, B_2, B_3) \mid A_1 \dot\cup A_2 \dot\cup A_3 = A, B_1 \dot\cup B_2 \dot\cup B_3 = B, x \in A_3, \#A_2 = \#B_3, \#A_3 = \#B_2\} \\
&= 2 \cdot \sum_{\substack{A_3 \subset A \\ B_2 \subset B \\ \#B_2 = \#A_3 \\ x \in A_3}} \#\{(A_1, A_2, B_1, B_3) \mid A_1 \dot\cup A_2 = A \setminus A_3, B_1 \dot\cup B_3 = B \setminus B_2, \#A_2 = \#B_3\} \\
&= 2 \cdot \sum_{k=0}^{n} \sum_{\substack{A_3 \underset{k}{\subset} A \\ B_2 \underset{k}{\subset} B}} k \cdot \#\{(A_1, A_2, B_1, B_3) \mid A_1 \dot\cup A_2 = A \setminus A_3, B_1 \dot\cup B_3 = B \setminus B_2, \#A_2 + \#B_1 = n - k\}
\end{aligned}$$

Putting $X = A_2 \dot\cup B_1$ (and conversely $A_2 = X \cap A$, $B_1 = X \cap B$), this is

$$\begin{aligned}
N &= 2 \cdot \sum_{k=0}^{n} \sum_{\substack{A_3 \underset{k}{\subset} A \\ B_2 \underset{k}{\subset} B}} k \cdot \#\{X \subset (A \setminus A_3) \dot\cup (B \setminus B_2) \mid \#X = n - k\} \\
&= 2 \cdot \sum_{k=0}^{n} \sum_{\substack{A_3 \underset{k}{\subset} A \\ B_2 \underset{k}{\subset} B}} k \cdot \binom{2(n-k)}{n-k} \\
&= 2 \cdot \sum_{k=0}^{n} \binom{n}{k}^2 \cdot k \cdot \binom{2(n-k)}{n-k} \\
&= 2 \cdot \sum_{k=0}^{n} (n-k) \cdot \binom{n}{k}^2 \cdot \binom{2k}{k} && \text{by substitution } k \mapsto n-k
\end{aligned}$$